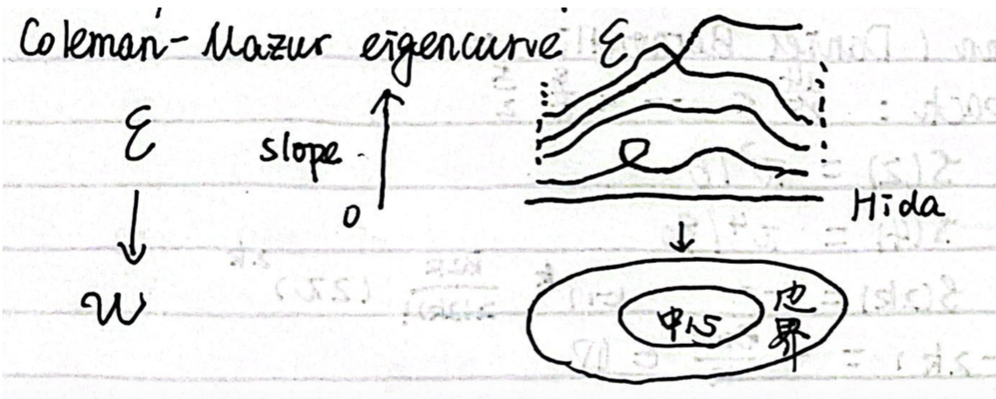
§ 1. Introduction

The eigencurve is a rigid analytic (or, if you prefer, adic) space that parametrizes p-adic systems of Hecke eigenvalues.
A cartoon of the construction: ones put together all modular forms of (tame level N) according to p-adic distance and then take completion. Rough picture in my head —



Where,

- $\mathcal{E}$ denotes the eigencurve;
- $\mathcal{W}$ the weight space, which is the union of $(p-1)$ open unit discs;
- "Slope" means the p-adic valuation of a special Hecke operator U_p. The slope zero (aka ordinary) part is the famous Hida family;
- The map weight map $w : \mathcal{E} \to \mathcal{W}$ is locally quasi-finite and flat, so in particular $\mathcal{E}$ is of equidimension 1, justifying its name.

Since its introduction by Coleman and Mazur in 1996, the eigencurve (and its variants) has been the focus of significant interest over the past two decades. As you can see already, there is some mystery going on in this picture above. For instance,

- it is not known whether the number of connected components of the eigencurve is finite or not
- the behavior of the eigencurve near its boundary and at the center seems very different
- ...
  Let me just remark that there is remarkable progress regarding these

questions very recently, see e.g. [LTXZ25].

Today, I would like to talk about two very specific local questions. Namely, I start with a classical new form of weight 1 and view it as a point x on the eigencurve and then ask:

1. Is the eigencurve smooth at x, and moreover?
2. Is the weight map étale at x?
   In the case where f is "p-regular", the answer two both of these questions are YES, given in a beautiful and seminar paper of Bellaïche and Dimitrov.

$\S\, 2$. Local geometry of the eigencurve

To give you a taste of what they did in [BD16], let me put up a table.

Automorphic side	Galois side
p-adic modular forms	Galois representations
eigencurve	families of Galois (pesudo-)reps
nbhd of a point	deformations of Galois reps
tangent space	tangent space
fiber over a wt	fixing determinant

The Galois side is easier to understand because we have more algebraic tools available —

- Galois cohomology
- Class field theory
- p-adic transcendental number theory (Baker--Brumer)
  e.g.
- we may express tangent spaces of Galois deformation problems in terms of Galois cohomology
- $\mathrm{H}^1(K, \overline{\mathbb{Q}}_p) = \mathrm{Hom}(G_K, \overline{\mathbb{Q}}_p) = \mathrm{Hom}(G_K^{\mathrm{ab}}, \overline{\mathbb{Q}}_p)$, so CFT pops up
- $\mathrm{Hom}(\mathcal{O}_{K,v}^{\times}, \overline{\mathbb{Q}}_p)$ whose elements involve the p-adic logarithm, and p-adic TNT helps one to control the dimension of these things

Now, their main strategy can be summarized as follows:

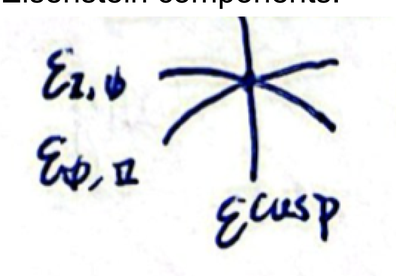
1. Introduce certain deformation problem $\mathcal{D}$ for ρ, which is representable by some local Noetherian $\overline{\mathbb{Q}}_p$-algebra R. By construction,

$$R \twoheadrightarrow T,$$

where T is the completed local ring of $\mathcal{E}$ at x.
2. Compute the tangent space to $\mathcal{D}$ (which corresponds to the tangent space of $\mathcal{E}$ at x) and show that $\dim t_{\mathcal{D}} = 1$. Conclude that $R \xrightarrow{\sim} T$ and $\mathcal{E}$ is smooth at this point.
3. A similar argument works for étaleness.

$\S$ 3. Further developments and applications

I haven' told you why people care about these questions, so let me do it now. A striking extension of the story comes from the work of Betina, Dimitrov, and Pozzi [BDP21]. They investigated the following case beyond the p-regular case, starting with an evil (critical) Eisenstein series $E(\mathbb{1}, \phi)$ with Dirichlet character ϕ. $E(\mathbb{1}, \Phi)$ is not cuspidal in the classical sense but becomes one in the p-adic world. So it naturally lives on both the cuspidal part as well as, in fact, two Eisenstein components.



[BDP22] showed in particular that the cuspidal component intersects transversally with the Eisenstein component at this point, and then deduced a new proof of the celebrated Ferrero--Greenberg theorem - the -adic L-function has a simple zero at $s = 0$. With more efforts, they were also able to recover the Gross--Koblitz formula

$$L_p'(\phi\omega_p, 0) = -\mathcal{L}(\phi)L(\phi, 0),$$

relating the first order derivative of p-adic L-function L_p at $s = 0$ with the complex L-value at the same point. Here, $\mathcal{L}(\phi)$ is the so-called $\mathcal{L}$-invariant associated to ϕ, which involves the p-adic logarithm of certain p-adic units, and you can imagine that p-adic transcendental NT plays an important role as well.