Geomtery and Algebra of (B, N)-pairs

Part I. Excursion to buildings

Reference:

- For Part I and some parts of II, I borrow heavily from Petra Schwer's mini course on buildings
- For the tree example, see Serre's notes on tree; Bill Casselman's notes "The Bruhat--Tits Tree of SL(2)" as well as other essays such as "Geometry of the tree" on his webpage

§ 1 Examples

The Fano plane

Given a finite field $\mathbb{F}_q$ of order q, consider the finite project plane $\mathbb{P}^2(\mathbb{F}_q)$; or in a more elementary language

- $G := \mathrm{GL}_3(\mathbb{F}_q)$ or $\mathrm{SL}_3(\mathbb{F}_q)$;
- $V := \mathbb{F}_q^3$;
- $\mathcal{P} := \{1-\text{dim. subspaces of } V\}$ called **points**;
- $\mathcal{L} := \{2-\text{dim. subspaces of } V\}$ called **lines**.
  We say a point P is **incident** to a line l if (as linear subspaces) $P \subset l$.

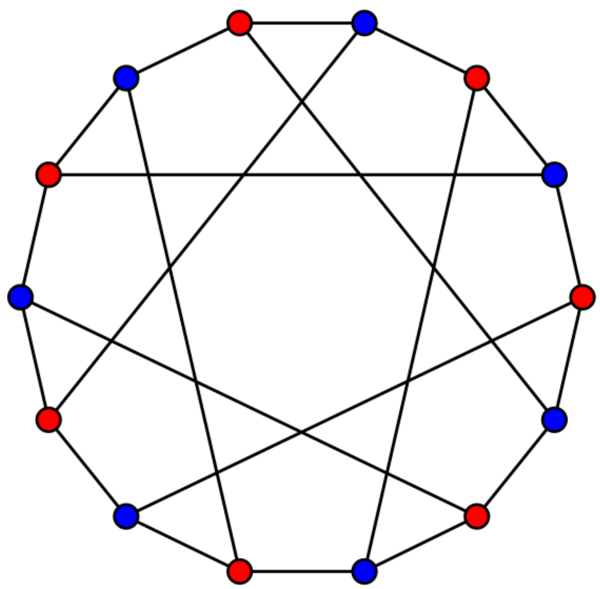
Note that

- $|\mathcal{P}| = |\mathcal{L}| = q^2 + q + 1$
  - there are $(q^3-1)/(q-1)$ distinct vectors up to scalar;
- Each point is contained in $(q+1)$ distinct lines (I am not sure if the following is the shortest way of seeing this)
  - the number of distinct lines can be computed also as follows: note that the number of 2 linear independent vectors in V is $(q^3-1)(q^3-q)$;
  - since elements in $\mathrm{GL}_2(\mathbb{F}_q)$ corresponds to change of bases, we quotient the number we got by $\#\mathrm{GL}_2(\mathbb{F}_q) = (q^2-1)(q^2-q)$ to obtain q^2+q+1 distinct lines,

- similarly, the number of lines *not* passing through a given point is $(q^3-q)(q^3-q^2)/\#\mathrm{GL}_2(\mathbb{F}_q) = q^2)$, where the number $(q^3-q)(q^3-q^2)$ is linearly independent vectors to a given vector
- Each line contains $(q+1)$ distinct points
  - given a line $l = \langle v_1, v_2 \rangle$, points on l are of the form $\lambda v_1 + \mu v_2$ for some λ and μ, and so counting distinct points is the same as counting points on the projective line $\mathbb{P}^1(\mathbb{F}_q)$

The **incidence graph** Δ_q of the projective plane is defined as follows:

- vertices correspond to elements of $\mathcal{P} \sqcup \mathcal{L}$;
- a pair (P, l) forms an edge iff P is incident to l.
  For instance, when $q = 2$ (coefficients for lazy mathematicians) we have the following picture,

  known as the "Heawood graph" (borrowed from Wikipedia).

Let $\{e_1, e_2, e_3\}$ be the standard basis o V, then points can be listed as

$$\mathcal{P} = \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3\}$$

and $\mathcal{L}$ are spans of pairs of distinct points.

The graph Δ_2 is a union of hexagons (e.g. the one A_0 with vertices $e_1, \langle e_1, e_2 \rangle, e_2, \langle e_2, e_3 \rangle, e_3, \langle e_3, e_1 \rangle$), called **apartments** in the language of buildings. The edges are called **chambers**. Apartments are in 1-1 correspondence to choices of basis of V. In particular, the symmetric group S_3

acts on this apartment by permuting the elements of the basis. Let us call A_0 the **fundamental apartment**.

Connection between G and Δ_q?
As a matrix group $G = \mathrm{GL}_3(\mathbb{F}_q)$ (or $\mathrm{SL}_3(\mathbb{F}_q)$ acts on $V = \mathbb{F}_q^3$ by multiplication. Thus, G acts on Δ_q and permutes points as well as lines. In fact, G acts transitively on pairs (e, A) of a chamber in an apartment (such a condition is called being **strongly transitive**). Equivalently, this means that G acts transitively on apartments, and the stabilizer of an apartment acts transitively on chambers in this apartment.

Now, let us compute the stabilizer of e connecting e_1 and $\langle e_1, e_2 \rangle$: Note that

- $\mathrm{Stab}_G(e_1) = \begin{matrix} & & \\ 0 & & \\ 0 & & \end{matrix} =: P_2$

- $\mathrm{Stab}_G(\langle e_1, e_2 \rangle) = \begin{matrix} & & \\ & & \\ 0 & 0 & \end{matrix} =: P_1$

  are standard *Parabolic subgroups* of G, we see that

  $\mathrm{Stab}_G(e) = P_1 \cap P_2 =: B = \begin{matrix} & & \\ 0 & & \\ 0 & 0 & \end{matrix}$ is the standard *Borel subgroup*.

If we look at the stabilizers of the fundamental apartment A_0, then a computation shows that

- the point-wise stabilizer is $T =$, the *maximal torus* of G; while
- the set-wise stabilizer is the *monomial subgroup* N of G, which normalizes T
.

Observations

- We have $\mathcal{L} \ {}^{1-1} \ G/P_1$, $\mathcal{P} \ {}^{1-1} \ G/P_2$, and
$\mathcal{C} := \{\text{chambers in } \Delta_q\} \ {}^{1-1} \ G/B$. In other words, all simplices of Δ_q are uniquely labeled by a left coset of P_1, P_2, or B.
- Edges gB and B share a P_i-vertex iff $gP_i = P_i$.
- $W := N/T \simeq S_3$, and in particular this is a "Coxeter group" - which I will define this in a moment. W is generated by $s_1 = \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$ and $s_2 = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$
satisfying $(s_1 s_2)^3 = \mathrm{id}$.
- $P_i = B \sqcup B s_i B$ for $i = 1, 2$.
- $G = \sqcup_{w \in W} BwB$ (the Bruhat decomposition).
The B, N above is an instance of a *(spherical)* (B, N)*-pair*.

A tree for SL_2

Let K be a non-Archimedean local field, $\mathcal{O}$ its valuation ring, ϖ a uniformizer, and $k = \mathcal{O}/\varpi\mathcal{O}$ the residue field. Let $G = \mathrm{SL}_2$.

By a **lattice** L in $V = K^2$ we mean a finitely-generated $\mathcal{O}$-submodule of V that spans V as a vector space (such a module is free of rank 2). Let X be the set of lattices in V up to homothety (that is, $L \sim L'$ if $\exists\, k \in K^\times$ such that $L' = kL$), and for a given lattice L we denote by $[L]$ its homothety class.

We now define a graph Δ as follows:

- vertices correspond to elements in X;

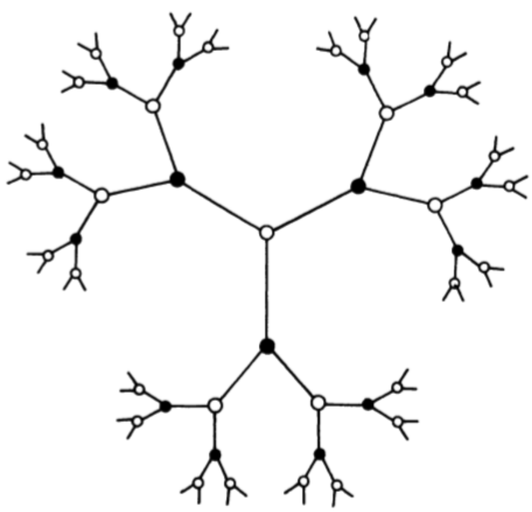
- an edge between $[L]$ and $[L']$ exists if there exists some representatives L and L' such that $L' \subset L \subset \varpi L'$, or equivalently $L/L' \simeq k$.
  - By the structure theory of modules over a PID, for any two lattices L and L' there exists an $\mathcal{O}$-basis $\{e_1, e_2\}$ of L and integers a, b such that $\{\varpi^a e_1, \varpi^b e_2\}$ is a basis for L'. The integers a, b does not depend on the choice of bases for L and L'. In which case, $L/L' \simeq (\mathcal{O}/\varpi^a\mathcal{O})\ \ (\mathcal{O}/\varpi^b\mathcal{O})$.
  - In fact, this allows us to define a "distance" on this tree: $d(L, L') := |a - b|$.

Theorem (Structure of Δ).

If $\#k = q$, then the graph Δ is a $(q+1)$-regular tree without leaves.

Life is short, so we will skip the proof. My intuition comes from

In the case where $k \simeq \mathbb{F}_2$, it looks like:



(Where, black and white nodes correspond to a partition of the vertices into two classes such that vertices in the same class are an even distance apart.)

A line (roughly speaking, it is a path on the tree without going backwards that goes to infinity on both sides - see Casselman for a rigorous definition) in the tree above corresponds to a choice of a basis of V. For instance, fixing a basis

$\{e_1, e_2\}$ of V, then

$$\langle \pi^a e_1, e_2 \rangle, \quad a \in \mathbb{Z}$$

forms a line A_0. Again, let us call these edges "chambers"; lines "apartments", and A_0 the fundamental apartment. To me, this is very striking.

- One way to see this is to prove that every line "ends" with two points ℓ_0 and ℓ_∞ in $\mathbb{P}^1(K)$, and if x_i is a generator of ℓ_i, the nodes of the apartments are

  $$\langle \varpi^m x_0, x_\infty \rangle$$

  for $m \in \mathbb{Z}$ (Corollary 7.2 of Casselman's "Geometry of the tree"). Why is that? Let us fix a lattice L consider first lattices L' that has distance n from L, that is, $L/L' \simeq \mathcal{O}/\varpi^n\mathcal{O}$. Since $\varpi^n L \subset L' \subset L$ and $L/\varpi^n L \simeq (\mathcal{O}/\varpi^n\mathcal{O})^2$, we are actually looking for rank 1 direct summands of $(\mathcal{O}/\varpi^n\mathcal{O})^2$, which are exactly elements of the projective line $\mathbb{P}^1(\mathcal{O}/\varpi^n\mathcal{O})$. Now, passing to infinity gives

  $$\mathbb{P}^1(\mathcal{O}) \simeq \lim \mathbb{P}^1(\mathcal{O}/\varpi^n\mathcal{O})$$

  and elements in $\mathbb{P}^1(\mathcal{O})$ is in 1-1 correspondence with $\mathbb{P}^1(K)$. **Here, I am generalizing the notion of projective spaces**: by $\mathbb{P}^1(\mathcal{O})$ I mean a rank 1 direct summand of $\mathcal{O}^2$ and similarly for $\mathcal{O}/\varpi^n\mathcal{O}$. See Casselman's "Geometry of the tree", section 2 and p. 72 of Serre.
- Another way to see this is to use the fact that $G(K)$ acts transitively on apartments of Δ, but it is kind of cheating because the fact we are using is far from obvious... I will refer the reader to section 4 of Casselman's "Geometry of the tree"

Observations

- $G(K)$ acts on the tree Δ. And again, this action is strongly transitive, that is, transitively on pairs of a chamber contained in an apartment.
- The fundamental apartment A_0 is set-wise stabilized by the monomial subgroup $N(K)$, which is generated by $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the maximal torus $T(K)$; it is point-wise stabilized by $T(\mathcal{O})$.
  - s acts on it by reflection and $T(K)$ acts on it by translation.
- $W = N(K)/T(\mathcal{O})$ is a Coexter group.

- Stabilizers of the fundamental chamber, that is, the edge between $\langle e_1, e_2\rangle$ and $\langle e_1, \varpi e_2\rangle$, is given by the *Iwahori subrgoup* $I = \left\{\begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \varpi & \mathcal{O}^\times \end{pmatrix}\right\}$, which is the pre-image under the reduction map $\mathcal{O} \twoheadrightarrow k$ of the Borel subgroup $B(k)$
.
The pair $(I, N(K))$ is an (*affine*) (B, N)*-pair for* $\mathrm{SL}_2(K)$.

Remarks:

- One can generalize both stories to SL_n...
- There are much more interesting things to discover, e.g., one may ask how does unipotent matrices $N(\varpi^m) = \begin{pmatrix} 1 & \varpi^m\mathcal{O} \\ & 1 \end{pmatrix}$ acts on Δ, etc. and this would lead one to an geometric interpretation of the Iwasawa decomposition
$$G = NAK.$$
This is analogous to the real case - important for studying representations and automorphic forms on SL_2, etc.

$\S$ 2 One formalism of buildings

Both of the graphs Δ above are examples of buildings! Roughly speaking, buildings are made up of apartments, a.k.a. *thin buildings*, which correspond to Coxeter groups. Hence, before giving a definition of buildings we recall the very basics of Coxeter groups and complexes.

Coxeter groups and complexes

A **Coxeter group** W is a finitely-generated group given by a presentation,

$$W = \langle s_1, \dots, s_n : s_i^2 = e, (s_i s_j)^{m_{ij}} = e\rangle,$$

where $m_{ij} \geq 2$ for $i \neq j$ and can be ∞ (meaning that there is no relation between s_i and s_j). Denote by S the set of generators $s_1, \dots, s_n$. We call the pair (W, S) a **Coxeter system**.

First examples: finite reflection groups (including Weyl groups), D_∞, $\mathrm{GL}_2(\mathbb{Z})$, etc.

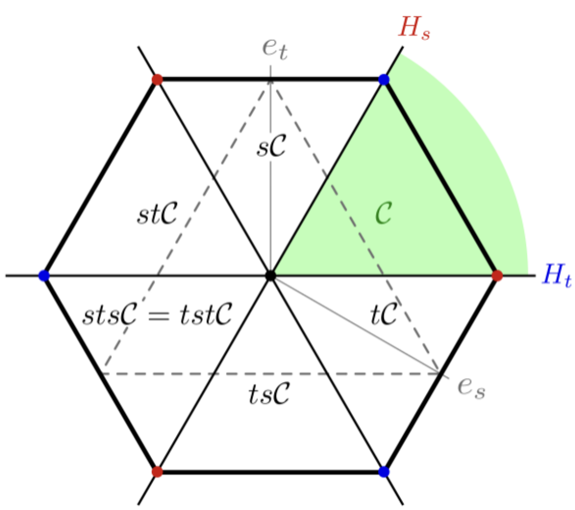
Facts

- Coxeter groups come in three classes: finite (spherical), affine, and indefinite. Finite Coxeter groups are classified in terms of their Coxeter diagrams.
- Affine Coxeter groups split as semi-direct products of a spherical Coxeter group and a translation group isomorphic to $\mathbb{Z}^n$, $n = |S|$.
- All Coxeter groups are linear, i.e., admit a faithful representation $\rho : W \to \mathrm{GL}_n(V)$ where V is a real vector space of dimension $n = |S|$.
- All Coxeter groups come with a simplicial complex, called the **Coxeter complex**, on which they naturally act.

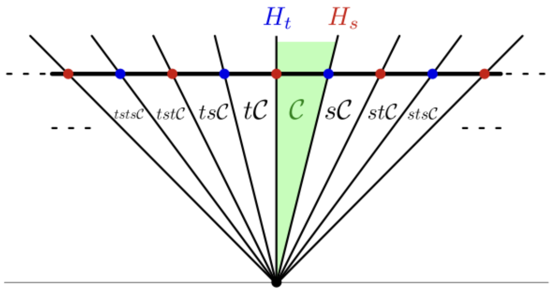
Construction of the Coxeter complex: Given a Coxeter system (W, S), consider

- special subgroups $W_I = \langle s_i, i \in I \rangle < W$ for some I S (can be taken as $\emptyset$), and
- special cosets wW_I for some $w \in W$, I S.
  Let $\Sigma = \Sigma(W, S)$ be the poset of special cosets ordered by reverse inclusion. It is indeed a simplicial complex (Abramento & Brown's book "Buildings: Theory and Applications" Theorem 3.5); When W is finite (spherical), Σ is isomorphic to some complex associated to a finite reflection group, and hence it is a simplicial complex triangulating a sphere.

Here are two examples.



The Coxeter complex for S_3



The Coxeter complex for D_∞

These two pictures are borrowed from Wikipedia, and in the talk I did them in a more careful way.

Buildings

Now we are ready to define buildings.

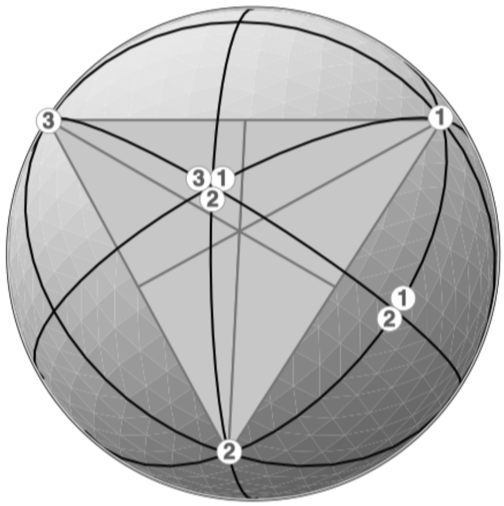
Definition (Building).

A building is a simplicial complex $\Delta \neq \emptyset$ together with a collection of subcomplexes $A \in \mathcal{A}$, called apartments, such that the following are satisfied:
(B0) Each apartment is a Coexter complex;
(B1) For any two simplices $a, b \in \Delta$, there exists an apartment $A \in \mathcal{A}$ such that $a, b \in A$;
(B2) If A and A' are two apartments containing a and b, then there exists an isomorphism $A \to A'$ fixing a and b pointwise.

Here is the canonical example of a building: **Flag complexes**. Let K be a field and let $\Delta = \Delta(K^n)$ be the abstract simplicial complex whose vertices are the non-zero proper subspaces of the vector spaces k^n and whose simplices are the chains

$$V_1 \;\; V_2 \;\; \cdots \;\; V_r$$

of such subspaces. Every simplex is contained in a subcomplex, called an apartment, which is isomorphic to the Coxeter complex associated with the symmetric group S_n. To find such an apartment, choose a basis $e_1, \ldots, e_n$ of K^n such that every subspace V_i that occurs in is spanned by some subset of the basis vectors. We then get an apartment containing by taking all simplices whose vertices are spanned by subsets of the basis vectors. The figure below shows an apartment for the case $n = 4$.



Where, the labels on the vertices indicate which basis vectors span the corresponding subspace.

Part II. Geometric side of (B, N)-pairs

§ 1 Buildings and (B, N)-pairs

We have seen the close connection between SL_n and buildings via examples in the previous section. Let us explore this further now.
Goal: Characterize the buildings Δ in terms of the subgroups B and N and vice versa.

Definition ((B, N)-pair).

A (B, N)-pair (or Tits system) for a group G is a pair (B, N) of subgroups of G such that the following hold:
(BN0) G is generated by $B \cup N$;
(BN1) $T := B \cap N$ is normal in N and the group $W := N/T$ has a generating system $S = \{s_1, \ldots, s_n\}$ such that
(BN2) For any $w \in W$ and $s_i \in S$,

$$Bs_iB \cdot BwB \subset BwB \cup Bs_iwB;$$

(BN3) For any $s_i \in S$, $s_iBs_i^{-1} \neq B$.

Remarks:

- S is uniquely determined by the axioms above.
- One can show that W is a Coxeter group.
- Though the axiom for (BN2) seems not symmetric, it turns out that we also have

$$BwB \cdot Bs_iB \subset Bs_iwB \cup BwB.$$

How to construct a building from a (B, N)-pair?

> **Theorem** (Tits).
>
> *Let G be a group with a (B, N)-pair (enough to assume (BN0) - (BN2)). Put $W := N/T$ with $T := B \cap N$. Then there exists a building $\Delta = \Delta(B, N)$ of type (W, S) with "nice" properties.*

What "nice" means here will become apparent as some hints towards the construction of Δ are provided:

- For $I \ \ S$, write $W_I := \langle s_i, i \in I\rangle$ as before. Define $P_I := {}_{w \in W_I} BwB$. These are subgroups of G (by axioms of (B, N)-pairs) and we will call the **standard Parabolic subgroup of type** I.
    - For instance, in the Fano plane example $P_{\{i\}}$ recovers P_i, and $P_\emptyset = B$.
    - It follows from axioms of (B, N)-pairs that these P_I's are distinct for different I's, and they are the only subgroups of G containing B. Moreover,

      $$P_I \cap P_J = P_{I \cap J}, \quad \langle P_I \cup P_J\rangle = P_{I \cup J}.$$

      In particular, P_I forms a poset under reversed inclusion.
- Consider the poset

  $$\{gP_I : I \ \ S, g \in G\}$$

  ordered by reverse inclusion. It turns out that this gives a realization of the building $\Delta = \Delta(B, N)$. Furthermore,
    - Chambers are 1-1 with G/B;

- Codimension-one faces are 1-1 with $G/P_{\{i\}}$, and more generally
- Codimension-k faces are 1-1 with G/P_I, where I runs over all subsets of S with cardinality k.

- If we pick the chamber c_0 corresponding to B as the fundamental chamber, then the fundamental apartment A_0 containing c_0 corresponds to N-orbits of P_I, which is

$$\{nP_I : I \ \ S, n \in N\} = \{wP_I : I \ \ S, w \in W\}.$$

  This is canonically isomorphic to the Coxeter complex associated to the Coxeter system (W, S).
- G acts strongly transitively on the system of apartments $\mathcal{A} = \{gA_0 : g \in G\}$.
- If (BN3) holds, then Δ is *thick*, meaning that every codimension-one face of a chamber is contained in at least 3 chambers.

From buildings to (B, N)-pairs

Conversely, given a building Δ with a group G acting on it. Suppose G acts transitively on the set $\mathcal{A}$ of apartments of Δ, and choose an arbitrary pair (c, A) of a chamber in an apartment. Remember A is isomorphic to the Coxeter complex $\Sigma(W, S)$ for some Coxeter group W. Put

- $B := \{g \in G : gc = c\}$;
- $N := \{g \in G : gA = A\}$; and
- $T := \{g \in G : g \,\text{fixes}\, A \,\text{pointwise}\} = B \cap N$.
  Then $T \ \ N$, because it is the kernel of the homomorphism $f : N \to W$ induced by the action of N on A. A standard argument in the theory of buildings also shows that f is surjective, and so $W \simeq N/T$ (See for instance Chapter 6 of Abramento and Brown. We have the following "converse" to Tits' theorem above:

Theorem ("Converse" to Tits' theorem).

With notations as above, the pair (B, N) satisfies (BN0)-(BN2). If Δ is thick, then (BN3) is also satisifed.

Remark

- Spherical buildings of rank $= |S| \geq 3$ are classified; they all arise from spherical (B, N)-pairs.
- Affine buildings of dimension ≥ 3 are classified; they all arise from affine (B, N)-pairs obtained via groups over non-Archimedean local fields. We call them **Bruhat--Tits buildings**.

$\S\ 2$ More on the Bruhat--Tits case

Let G be a semisimple linear group over a non-archimedean local field K. From

$$k = \mathcal{O}/\varpi\mathcal{O} \ \ \mathcal{O} \hookrightarrow K$$

we can actually get three (B, N)-pairs: For instance, when $G = \mathrm{SL}_n$ we have

- $(B(k), N(k))$, where $W \simeq S_n$ is spherical (the case $n = 3$ and $k \simeq \mathbb{F}_2$ recovers our Fano plane example), we get a building Δ_1;
- $(I, N(K))$, where $W \simeq S_n \ \ \mathbb{Z}^{n-1}$ is affine (when $n = 2$ we see $W = S_2 \ \ \mathbb{Z} \simeq D_\infty$, which is consistent with our tree example), we get a building Δ_2 which looks like
- $(B(K), N(K))$, where $W \simeq S_n$ is spherical again; we get a building Δ_3, apartments are hexagons, but there are infinitely many edges at every vertex. Very vague philosophy: Δ_1 is a "local picture of Δ_2", and Δ_3 can be though of as the boundary of Δ_2 at infinite.

$\S\ 3$ The Bruhat decomposition

From the (B, N)-pair axioms we may prove that G satisfies Bruhat decomposition:

Theorem 1 (Bruhat decomposition).

If G has a (B, N)-pair, then

$$G = \bigsqcup_{w \in W} BwB$$

in the spherical case; and

$$G = \bigsqcup_{w \in W} IwI$$

in the affine case.

Proof. 1). $G = \cup_{w \in W} BwB$. By (BN0) we see that $\forall\, g \in G$,

$$g = b_1 n_1 b_2 n_2 \cdots b_k n_k b_{k+1}$$

for some $b_i \in B$ and $n_i \in N$. This means that

$$g \in Bn_1Bn_2B \cdots Bn_kB = Bw_1Bw_2B \cdots Bw_kB.$$

Now, we may repeatedly apply (BN2) to prove that $g \in BwB$ for some B.
2). Disjointness. Our strategy is induction on

$$d := \min\{l_S(w), l_S(w')\}$$

where we assume that $w, w' \in W$ are such that $BwB \cap Bw'B \neq \emptyset$ (note that that actually implies $BwB = Bw'B$) and $l_S(w)$ denotes the length of w w.r.t. the set of generators S. WLOG, assume that $l_S(w') \leq l_S(w)$.

- The case $d = 0$ is easy because if $BwB = B$ then $w \in B$ and so $w \in B \cap N = T$, meaning that w is trivial in W.
- As for the induction step, let us write $w' = sw''$ such that w'' has length smaller than w' for some s. Then our assumption $BwB = Bw'B$ implies that $sw''B \subset BwB$ and hence (recall that s has order 2) $w''B \subset sBwB$. It follows that

$$Bw''B \subset BsB \cdot BwB \subset BwB \cup BswB.$$

  Since $Bw''B \cap BwB = \emptyset$ by induction hypothesis, $Bw''B = BswB$, and this implies $w'' = sw$ by induction hypothesis again, meaning $w' = sw'' = w$.

□

Geometrically, the double coset IwI (more generally BwB) can be viewed as the collection of chambers of Δ that are in the same I (resp. B) -orbit as $wI \in \mathcal{A}_0$. The Bruhat decomposition shows that there are every chamber lies in exactly one such an orbit, indexed by elements $w \in W$.
In fact, we can say more: that G acts on Δ strongly transitively implies this action is "Weyl-transitive", one equivalent description of which is that G acts transitively

on chambers of Δ and the stabilizer of a chamber c is transitive on on w-sphere

$$\{c' \in \mathcal{C} : (c', c) = w\}.$$

Where, is the so-called W-metric on Δ, which can be thought of as a W-valued "distance function" on Δ. If we take c to be the chamber corresponding to B, then B-orbits of c are in 1-1 correspondence with chambers that has distance w to c. In short, the Bruhat decomposition gives a partition of Δ into collections of chambers according to its distance to c. (There is an approach to buildings as W -metric spaces, and I found the expository paper "A (very short) introduction to buildings" by Brent Everitt a pleasant read.)

Part III. Back to algebra

In the last few weeks, starting from a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ we proved the existence of Chevalley basis $\{e_\alpha, \alpha \in \Phi; {}_i, 1 \leq i \leq l\}$ and used it to construct the adjoint Chevalley group G_{ad} for $\mathfrak{g}_K$ over an arbitrary field K. G_{ad} is defined as the group generated by

- $x_\alpha(t) = \exp(\mathrm{ad}(te_\alpha))$ for $\alpha \in \Phi$, as well as
- (χ) for $\chi \in \mathrm{Hom}(\mathbb{Z}\Delta, K^\times)$.
  In particular, we denoted last time to understanding important subgroups of G_{ad}: Denoting the root subgroups $X_\alpha := \langle x_\alpha(t), t \in K\rangle$, we studied
- The unipotent subgroups $U := \langle X_\alpha, \alpha \in \Phi^+\rangle$ and $V := \langle X_\alpha, \alpha \in \Phi^-\rangle$ - elements in here are unipotent transformations;
- $\langle X_\alpha, X_{-\alpha}\rangle$ - we proved that there exists a surjective homomorphism $\Phi_\alpha : \mathrm{SL}_2(K) \twoheadrightarrow \langle X_\alpha, X_{-\alpha}\rangle$ with kernel $\{1\}$;
- The diagonal group $H := \langle (\chi) : \chi \in \mathrm{Hom}(\mathbb{Z}\Delta, K^\times)\rangle$ - we showed that it is Abelian and normalizes every root subgroup X_α, which implies that $G' := [G, G] = \langle X_\alpha, \alpha \in \Phi\rangle$; and we denote $H' = G' \cap H$;
- The monomial subgroup $N := \langle H, n_\alpha, \alpha \in \Phi\rangle$, where $n_\alpha = \Phi_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \in \langle X_\alpha, X_{-\alpha}\rangle$ - we showed that elements n_α permutes the root spaces as well as the ${}_\alpha$'s and that

$$N/H \simeq H'/N' \simeq W = \langle w_\alpha, \alpha \in \Delta\rangle;$$

- The Borel subgroup $B := UH$ and $B' = UH'$ (they are groups because H (and H') normalizes U) - we have $U \cap N = \{e\}$, $B \cap N = H$, $B' \cap N' = H'$, as well as $B \cap V = B' \cap V = \{e\}$.

We didn't really cover the Borel subgroup so I will try to expand a bit this part later...

Theorem ((B, N)-pair for Chevalley groups).

The pair (B, N) (resp. (B', N')) is a (B, N)-pair for G (resp. G').

If one tries to check the axioms (BN0) - (BN3), the only non-trivial part is that

$$BsBBwB \subset BwB \cup BswB,$$

which requires a careful study of certain subgroups of G. Carter spent almost the whole section 8.1 on this.

Definition (Parabolic subgroup).

A parabolic subgroup of G is a subgroup that contains some conjugate gBg^{-1} of the Borel subgroup B.

Theorem (Classification of parabolic subgroups).

Let G be a group with a (B, N)-pair. Then

1. *The subgroups $P_I = BW_IB$ as defined above are the only subgroups of G that contains B.*
2. *The P_I's are distinct for different I's. Moreover,*

$$P_I \cap P_J = P_{I \cap J}, \quad \langle P_I \cup P_J \rangle = P_{I \cup J}.$$

This follows purely from (B, N)-pair axioms, which relies on more detailed results concerning the double coset multiplication (BN1), proved for Chevalley groups in section 8.2 of Carter. It should not be surprising that these result would reflect certain geometric properties of the building attached to (B, N). Interested readers are kindly referred to Chapter 6 of Abramento and Brown.