Automorphic forms as bridge and gap

I. Two interesting problems

A. The Banach--Ruziewicz problem.

Question: Is the Lebesgue measure the *unique* finitely additive, rotation-invariant, probability measure on Lebesgue measurable sets of $\mathbb{S}^n$?

- Assuming "countable additive" instead of "finitely additive", then YES
- Hausdorff--Banach--Tarski paradox: For $n \geq 2$, $\mathbb{S}^n$ can be decomposed into finitely many pieces, from which one can reconstruct two copies of $\mathbb{S}^n$ by using only rotations from $\mathrm{O}(n+1)$ (impossible for $\mathbb{S}^1$)
- $n = 1$ NO (Banach, 1921) uses amenability of $\mathbb{S}^1$ as a discrete group
- $n \geq 4$ YES (Margulis, Sullivan, 1980s) uses Kazhdan's property (T)
- $n = 2, 3$ YES (Drinfeld, 1985) under the help of the Ramanjuan conjecture

B. Construction of expander graphs.

A finite connected regular graph $X = X(V, E)$ with a set V of n vertices and of degree k is called an (n, k, c)-**expander** if for every subset A of V,

$$|\partial A| > c\left(1 - \frac{|A|}{n}\right)|A|.$$

Here, $\partial A = \{y \in V : d(y, A) = 1\}$ is the boundary of A and d is the distance function on X. An **expander family** is a family of graphs $\{X_n\}_{n \geq 1}$, where

- X_n is a $(|X_n|, k, c)$-expander graph (k, c are fixed);
- $|X_n| \to \infty$ as $n \to \infty$.

Intuitively, graphs in an expander family have the following competing properties:

1. they are fairly sparse;
2. yet they are highly connected, and in fact highly "robust" in some sense.

By counting arguments, it is not hard to show that expander families exist. However, for practical purposes, it this very natural to ask the following --

Question: Can one construct explicitly expander families?

- Margulis (1975) uses Kazhdan's property (T); not necessarily Ramanujan
- Lubotzky--Phillips--Sarnark, and independently Margulis (late 1980s), under the help of the Ramanujan conjecture; Ramanujan graphs

You may ask why we care about expander families and their explicit constructions, e.g. "McMullen conjecture", brain (neurons as vertices and synapses as edges - intrinsic objects), transport / computer network, applications to error correcting codes ...

II. Automorphic representations and the Ramanujan conjecture

- G be a connected reductive group over $\mathbb{Q}$ (e.g. GL_2, $D^\times$ where D is a quaternion algebra)
- A_G be the identity component of the $\mathbb{R}$-points of the maximal split subtorus of the center of G
- $[G] := G(\mathbb{Q})\backslash G(\mathbb{A})/A_G$
- $L^2([G])$ regular representation of $[G]$ via right translation

Unlike what previous speakers did, I'm gonna define automorphic forms more or less properly, but by cheating:

Definition (Automorphic representations of G).

An *automorphic representation* of G is an irreducible subquotient of $L^2([G])$. It is *cuspidal* if it is a subquotient of $L^2_0([G])$.

Let K be a maximal compact subgroup of $G(\mathbb{A})$. For any automorphic representation (π, V) of G, write $V^{\mathbf{sm}}$ for its subspace of K-finite vectors. Denote $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))$.

Theorem (Bernstein--Harish-Chandra).

The subspace $V^{\mathbf{sm}}$ consists of "automorphic forms", and it is an irreducible admissible $G(\mathbb{A}_{\mathrm{fin}}) \times (\mathfrak{g}, K_\infty)$-module.

Fact (Flath Theorem): Every automorphic representation decomposes as a restricted tensor product $\pi = \otimes'_{p\leq\infty}\pi_p$.

Example: $G = \mathrm{GL}_1$

- $[G] = \mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}^+ \simeq \prod_p \mathbb{Z}_p^\times$ is abelian
- cuspidal condition is vacuous
- (cuspidal) automorphic representations of $G \leftrightarrow$ characters of $[G] \leftrightarrow$ primitive Dirichlet characters

Example: $G = \mathrm{GL}_2/\mathbb{Q}$

- $[G = \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathbb{R}^+]$ is what it is
- certain cuspidal automorphic representations of $G \leftrightarrow$ primitive new forms
- certain other -----//----- of $[G] \leftrightarrow$ primitive Maass cusp forms
  Here, "certain" (resp. "certain other") is essentially the condition on π_∞ of being (mock) discrete series / principal series (and possibly complementary series)

1. it is conjectured that Maass forms do not contribute to complementary series, which is equivalent to Selberg's $1/4$-conjecture
2. Harish--Chandra: if there's a god, then "complementary series" shouldn't exist

Example: $G = D_S^\times/\mathbb{Q}$ quaternion algebra ramified at the set of primes S

- Jacquet--Langlands correspondence: There exists an injection

  $$\text{JL}: \{\text{cusp. auto. reps. of } D^\times(\mathbb{A})\} \hookrightarrow \{\text{cusp. auto. reps. of } \mathrm{GL}_2(\mathbb{A})\}, \quad \pi \mapsto \pi'$$

  such that
  1. if $p \notin S$, $\pi'_p = \pi_p$;
  2. if $p \in S$, then π_p is a discrete series.
- in particular, if D is definite, then π'_∞ is discrete series, and I claim that π' is not in the complementary series of $\mathrm{GL}_2(\mathbb{Q}_p)$ for every finite prime p

I'll now explain why this claim is true.

Conjecture ((generalized) Ramanjuan conjecture).

Let $\pi = \prod_{p\leq\infty}$ be a cuspidal automorphic representation of G. If π is "generic", then every local component π_p is tempered. In other words, the matrix coefficients of π_p lie in the space of $L^{2+\epsilon}(G(\mathbb{Q}_p))$ for all $\epsilon > 0$.

What does this conjecture say? For $G = \mathrm{GL}_2$, every cuspidal automorphic representation is generic. So, conjecturally are two possibilities for the component π_∞:

- π_∞ is a (resp. mock) discrete series, so corresponding to a holomorphic cusp form f of weight $k \geq 2$ (resp. $k = 1$)
- π_∞ is a principal series, so corresponding to a Maass cusp form f (π is tempered iff f has eigenvalue $\geq 1/4$ of the Laplace operator. This coincides with the Selberg conjecture I mentioned above.)
  Moreover, in the case of $G = \mathrm{GL}_n$, if we let $L(\pi, s)$ be the L-function attached to the automorphic representation, then it admits an Euler product expression

  $$L(\pi, s) = \prod_{p<\infty} \left(1 + a_1(p)p^{-s} + a_2(p)p^{-2s} + \cdots + a_n p^{-ns}\right)^{-1}$$

  such that the local factor at a prime p is the reciprocal of a polynomial in p^{-s} of degree $\leq n$ with complex coefficients $a_1(p), \ldots, a_n(p)$. We have $a_n(p) \neq 0$ if π_p is unramified (which is the case for almost all p). And an unramified π_p is tempered iff all the roots of

  $$1 + a_1(p)X + \cdots + a_n(p)X^n$$

  have the same absolute value.

Now, it is clear that my claim follows from Deligne's proof of the Weil conjecture.

III. Applications to the two problems

For both Part A and Part B, I’ve written a long version followed by a short version. You can safely skip the long version if you prefer — it just contains more detailed background / explanation.
However, if you decide to read the long version, please note that it is meant to be

read together with the short version; the two parts combined form the complete discussion.

A. The Banach--Ruziewicz problem

--- Long version ---

Fell topology, Kazhdan property (T), and amenability

- G locally compact group
- $\hat{G}$ the set of equivalent classes of unitary representations of G

If ρ, π two unitary representations of G, we say that ρ is **weakly contained** in π, denote by $\rho \prec \pi$, iff every diagonal matrix coefficient of ρ is a limit, uniformly on compact subsets of G, of those of π.

Let (π, V) be a unitary representation of G. For any compact subset K of G, > 0, and $v \in G$ of norm one. We define open neighborhoods of (π, V) as

$$W(K, , v) = \{(\rho, W) \in \hat{G} : \exists\, w \in W \text{ such that } |w| = 1, |\langle v, \rho(g)v\rangle - \langle w, \sigma(g)w\rangle| <$$

The **Fell topology** is the topology determined $W(K, , v)$. By definition,

$$\rho \prec \pi \Leftrightarrow \pi \in \overline{\{\rho\}}.$$

We say that

- G has **Kazhdan property (T)** iff the trivial representation of G is an isolated point of $\hat{G}$ with respect to the Fell topology.
- G is **amenable** iff the regular representation of G weakly contains the trivial representation.

Intuition? I have the weird impression that I heard from somewhere that Kazhdan defined this property (T) when playing table tennis (which makes sense since he uses the letter T). So to understand this properly you should perhaps play table tennis.

Key Fact: Let Γ be a discrete group, and let $\Gamma \to \mathrm{SO}(n+1)$ be a homomorphism. Consider the representation $L^2(\mathbb{S}^n)_0$ of Γ. If $\mathbb{1}$ $L^2(\mathbb{S}^n)_0$, then the answer to the Banach--Ruziewicz problem is yes.

It is proved by an awful lot of analysis, which I will kindly skip for you since the only analysis I like is p-adic analysis.

Remark on the $n = 1$

- Tarski: Suppose a group G acts on some space X. Then there exists a finitely additive G-invariant probability measure on all subsets of X iff X is not G-paradoxical.
- $\mathbb{S}^1$ is amenable implies that it is not $\mathrm{SO}(2)$-paradoxical.
- Hence, it is reasonable to believe (at least for me) that if we only look at Lebesgue measurable sets there should be a lot of finitely additive, rotation invariant measures. That is, the answer should be no.

The case of $n \geq 4$

- Margulis & Sullivan constructed of finitely generated dense subgroups $\Gamma < \mathrm{SO}(n+1)$ with property (T) for $n \geq 4$, and from there it is easy to see that Banach--Ruziewicz problem has positive answers in those cases.
- In fact, Margulis proved that, any real compact simple Lie group G which is not locally isomorphic to $\mathrm{SO}(n)$ for $n = 2, 3$ or 4 admits a finitely generated dense subgroup with property (T). As a consequence, the Haar measure on G is the only G-invariant finitely additive probabilistic measure defined on the Haar measurable sets of G.

--- Shorter version ---

Recall key fact: Let Γ be a discrete group, and let $\Gamma \to \mathrm{SO}(n+1)$ be a homomorphism. Consider the representation $L^2(\mathbb{S}^n)_0$ of Γ. If $\mathbb{1}$ $L^2(\mathbb{S}^n)_0$, then the answer to the Banach--Ruziewicz problem is yes.

Let D be a definite quaternion algebra over $\mathbb{Q}$, so $D^\times(\mathbb{R})/A_{D^\times} \simeq \mathrm{S}(2)$. And

- for $n = 2$, it is classical that $\mathrm{S}(2)$ is a 2-cover of $\mathrm{SO}(3)$ (or, one can see this via conjugation by unit quaternions);
- for $n = 3$, right translation by unit quaternions yields a map $D^\times(\mathbb{R})/A_{D^\times} \to \mathrm{SO}(4)$.
  In both cases,

$$D^\times \hookrightarrow D^\times(\mathbb{R}) \to \mathrm{SO}(n+1).$$

Drinfeld's argument: Suppose $\mathbb{C} \prec L^2(\mathbb{S}^n)_0$. Since $D^\times(\mathbb{R})/A_{D^\times}$ acts transitively on $\mathbb{S}^n$, we have

$$L^2(\mathbb{S}^n)_0 \hookrightarrow L^2\left(D^\times(\mathbb{R})/A_{D^\times}\right).$$

It follows that $\mathbb{1} \prec \operatorname{Ind}_{D^\times}^{D^\times(\mathbb{A}_{\mathrm{fin}})} L^2\left(D^\times(\mathbb{R})/A_{D^\times}\right)$. This representation is a subrepresentation of $L^2[D^\times]$ on which $\mathrm{S}(2)$ acts trivially. As it weakly contains $\mathbb{C}$, for any $\epsilon > 0$ and for any p sufficiently large we can find an automorphic representation π of $D^\times(\mathbb{A})$ with

$$|a_p - (p+1)| < \epsilon.$$

This contradicts with temperedness of $\mathrm{JL}(\pi)_p$.

B. Construction of expander graphs

--- Longer version ---

A bit of spectral theory

For me, graphs will always be finite connected.

Let X be a k-regular graph. Let A be its adjacency matrix. Write $\lambda^0 \geq \lambda^1 \geq \cdots \geq \lambda^n$ for its eigenvalues

1. Every eigenvalue λ^i of A has $|\lambda^i| \leq k$.
2. k is an eigenvalue of multiplicity one.
3. $-k$ is an eigenvalue of A iff X is bipartite.
4. A family $\{X_n\}$ of k-regular graphs is expander for some c iff there exists some other $\delta = \delta(c) > 0$ such that $\lambda^1(X_n) \leq k - \delta$ for all n.

Rmk: There is a natural Laplacian operator Δ on $L^2(X)$ of a finite regular graph which is a discrete analogue of the geometric Laplacian operator, and which turns out to be the same as $k\mathrm{Id} - A$.
Now, in other words, the above says in particular that, for a regular graph we have

$$0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n.$$

It is expander if $\lambda_1 \geq \delta$, meaning that there exists a spectral gap of δ.

Theorem (Alon--Boppana inequality).

For a family of $\{X_n\}$ *k-regular graphs with* $|X_n| \to \infty$ *as* $n \to \infty$,

$$\lim_{n\to\infty} \lambda_1(X_n) \geq 2\sqrt{k-1}.$$

A family $\{X_n\}$ of k-regular graphs with $|X_n| \to \infty$ as $n \to \infty$ is called **Ramanujan** if it enjoys the extremal property that

$$\lambda_1(X_n) \leq 2\sqrt{k-1}$$

for every n.

Why is such a family called Ramanujan? Let me give you a hint by mentioning the following

Fun fact: The Ihara zeta function of a k-regular graph X satisfies the Riemann hypothesis iff X is Ramanujan.

--- Shorter version---

Theorem (Lubotzky--Phillips--Sarnak).

There are infinitely many discrete torsion-free co-compact subgroups Γ *of* $\mathrm{PGL}_2(\mathbb{Q}_p)$ *such that* $X_\Gamma := \Gamma\backslash\mathcal{B}_{n,F}$ *are Ramanujan graphs. Here,* $\mathcal{B}_{n,F}$ *denotes the Bruhat--Tits building associated to* $\mathrm{GL}_2(\mathbb{Q}_p)$.

Construction:

- D definite quaternion algebra unramified at p
- as a toy model, consider

$$\begin{aligned} X &:= D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A})/D^\times(\mathbb{R}) \prod_{p<\infty} D^\times(\mathbb{Z}_p) \\ &= D^\times\left(\mathbb{Z}\left[1/p\right]\right)\backslash D^\times(\mathbb{Q}_p)/D^\times(\mathbb{Z}_p) \\ &= D^\times\left(\mathbb{Z}\left[1/p\right]\right)\backslash \mathrm{GL}_2(\mathbb{Q}_p)/\mathrm{GL}_2(\mathbb{Z}_p) \end{aligned}$$

where the second row follows from strong approximation theory, and the last is simply by unramifiedness at p

  - The third expression gives the graph structure of X as a quotient of the infinite $k := (p+1)$-regular tree
  - The first expression of X allows us to view functions on vertices of X as automorphic forms on the quaternion algebra
- under JL, non-constant automorphic forms $\leftrightarrow$ certain cusp forms of weight 2
- adjacency matrix on X, viewed as an operator restricted to non-constant functions $\leftrightarrow$ Hecke operator T_p
- T_p Hecke eigenvalue $\leftrightarrow$ λ_1 of X
- Ramanjuan conjecture implies that $\lambda_1 \leq 2\sqrt{p} = 2\sqrt{k-1}$ and hence X is Ramanujan