Talk 7. The Coleman-Mazur eigencurve I: automorphic side

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<u>Plan</u>:

1) Overconvergent modular forms
2) The Coleman-Mazur eigencurve
3) *Some open questions

§1. Overconvergent modular forms

I. <u>Modular curves</u>

Fix $N \geq 5$ & $p \nmid N$.

- $X = X(\Gamma_1(N) \cap \Gamma_0(p)) \,/\, \mathbb{Z}[1/N]$ w/ R-pts given by ($\mathbb{Z}[1/N]$-alg.)

$$X_{\mathbb{Z}[1/N]} = \{\text{iso. cls. of } (E/\operatorname{Spec} R,\ H,\ \eta_N)\}$$

where

- $E/\operatorname{Spec} R$ generalized ell. curve
- $H \subset E[p]$ fin. loc. free rk. p subgp sch.
- $\eta_N : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]$ pt of exact order N

proper flat

usually easy to keep track of, so ignore...

- $Y = X(\Gamma_1(N)) / \mathbb{Z}[\frac{1}{N}]$ — " —

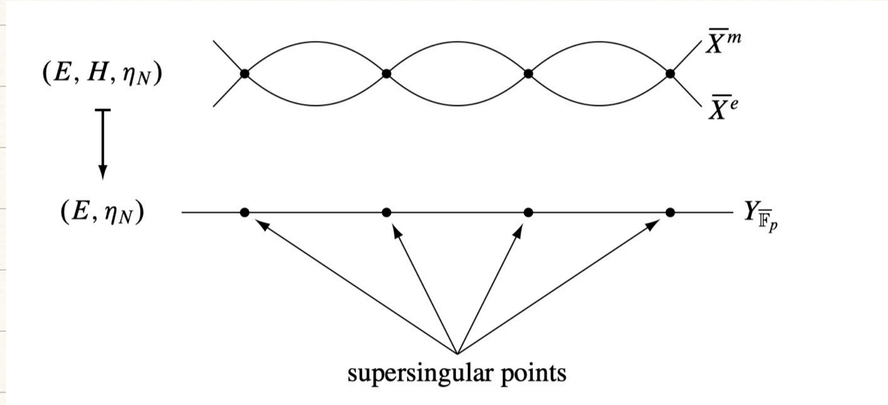
$$Y_{\mathbb{Z}[1/N]} = \{ \text{— " —} \ (E/\operatorname{Spec} R, \eta_N)\}$$

- ω = automorphic line bundle

$\underbrace{H^0(X(\Gamma)_{\mathbb{Q}}, \omega^k)}_{\text{v.s.}/\mathbb{Q}} \otimes \mathbb{C} \simeq M_k(\Gamma)$ ← classical def. of mod. forms

$\rightsquigarrow$ AG theory of mod forms mod p / p-adic

- $X_{\mathbb{Z}_p}$, $Y_{\mathbb{Z}_p}$ extension of scalars to $\mathbb{Z}_p$
  - sm. over $\mathbb{Z}_p$
- $X_{\mathbb{F}_p} \to Y_{\mathbb{F}_p}$ (geometric) special fibers



- X^m, X^e two open subschemes of $X_{\overline{\mathbb{F}}_p}$ corresp. to the locus where H is (étale-loc.) iso. to μ_p or $\mathbb{Z}/p\mathbb{Z}$, resp.
  - * $\bar{X}^m$, $\bar{X}^e$ irr. components of $X_{\overline{\mathbb{F}}_p}$;
  - * $\bar{X}^m \cap \bar{X}^e$ = ss pts;
  - * $(E, H) \mapsto E$ induces
    $\bar{X}^m \xrightarrow{\sim} Y_{\overline{\mathbb{F}}_p}$, $\bar{X}^e \to Y_{\overline{\mathbb{F}}_p}$ pure insep. map of deg d.
    and
  - * $(E, H) \mapsto E/H$ induces
    $\bar{X}^e \xrightarrow{\sim} Y_{\overline{\mathbb{F}}_p}$, $\bar{X}^m \to Y_{\overline{\mathbb{F}}_p}$ — " —

II. Overconvergent (OC) forms

Notation:

$(X_{\mathbb{Q}_p}^{an} = \mathcal{X}?)$

- $X \rightsquigarrow \mathfrak{X}$ = formal completion along $X_{\mathbb{F}_p} \rightsquigarrow \mathcal{X} = (\mathfrak{X})^{ad}_{\mathbb{Q}_p}$
  ⌜ loc. $\mathfrak{X}$ is covered by Spf R top. of fin. type / Spf $\mathbb{Z}_p$, i.e., quotient of $\mathbb{Z}_p\langle x_1, \cdots, x_n \rangle$
  $(\mathfrak{X})^{ad}$ is covered by $\mathrm{Spa}(R, R)$ &
  $(\mathfrak{X})^{ad}_{\mathbb{Q}_p}$ — " — $\mathrm{Spa}(R[1/p], R)$
- $sp : |\mathcal{X}| \to |\mathfrak{X}| = |X_{\mathbb{F}_p}|$ specialization map
  ⌜ $x \mapsto \{r \in R : |r|_x < 1\}$ ⌟
- Y, $\mathfrak{Y}$, $\mathcal{Y}$

- In $Y_{\mathbb{F}_p}$,
  - $Y^{ord}_{\mathbb{F}_p}$ ordinary locus, open affine
  - $Y^{ss}_{\mathbb{F}_p}$ ss locus
- $\mathcal{Y}^{ord} = sp^{-1}(Y^{ord}_{\mathbb{F}_p})$ open affinoid
- Over $Y^{ord}_{\mathbb{F}_p}$, $X_{\mathbb{F}_p} \to Y_{\mathbb{F}_p}$ (forget H) has a can. section, given by taking H to be the conn. part of $E[p]^\circ \simeq \mu_p$

  ⇝ extends to a section on formal completions ⇝

$$can: \mathcal{Y}^{ord} \to \mathcal{X}$$

iso. onto its image, denoted by $\mathcal{X}^{m,ord}$

⌜ If $C/\mathbb{Q}_p$ is complete & alg. cl., E/C ell. curve w/ ord. redn. then

$$can(E, \eta_N) = (E, H_{can}, \eta_N)$$

w/ H_{can} the canonical subgp. $\ker(E[p](\mathcal{O}_C) \to E[p](\mathcal{O}_C/p))$ coming from the p-torsion in the formal gp $\hat{E} \simeq \hat{\mathbb{G}}_m \Rightarrow$ $H_{can} \simeq \mu_p$ ⌟

- $Ha \in H^0(Y_{\mathbb{F}_p}, \omega^{p-1})$ Hasse inv.
  - mod p modular form of level 1, wt $p-1$
  - has simple zeros exactly at ss pts, so $V(Ha) = Y^{ss}_{\mathbb{F}_p}$ & $D(Ha) = Y^{ord}_{\mathbb{F}_p}$
  - q-expansion $= 1$.

⌜ Construction:

- $X \to S$ morphism of $\mathbb{F}_p$-schemes

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!\ Frob_{X/S}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\
& S \arrow[r, "Frob_S"] & S
\end{tikzcd}$$

"relative Frobenius"

e.g. $E/\mathbb{F}_p$ ell. curve

$E: y^2 = x^3 + ax + b$

$E^{(p)}: y^2 = x^3 + a^p x + b^p$

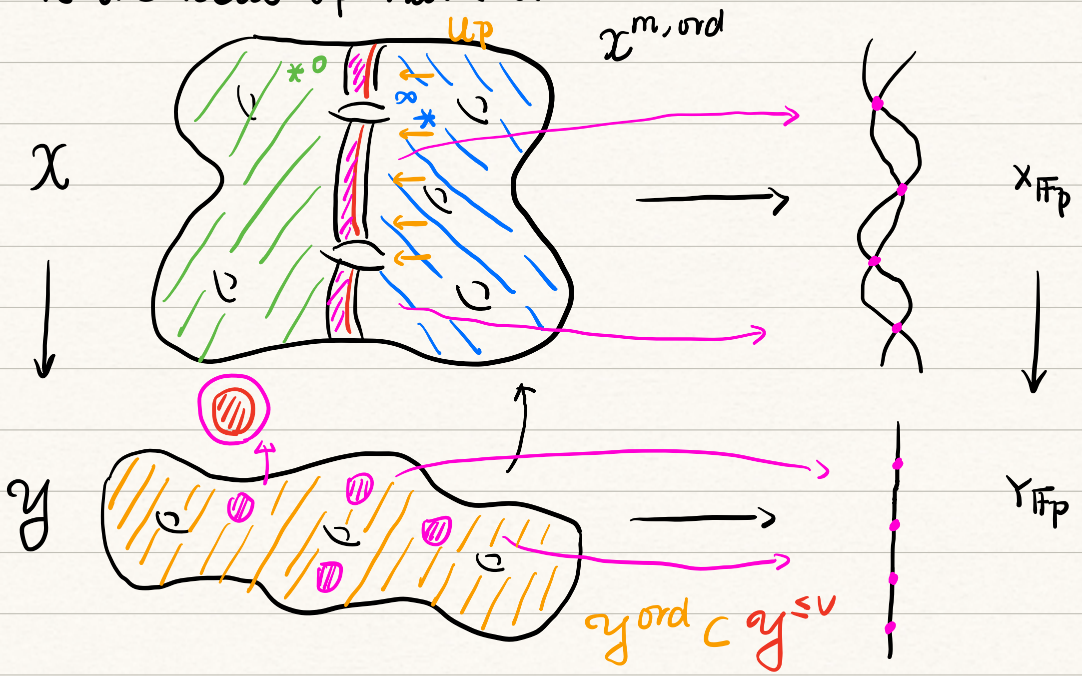
$Frob_{X/S}: (x, y) \mapsto (x^p, y^p)$

- $\mathrm{Ver}_{X/S} : X^{(p)\vee} \to X^{\vee}$ dual map $\qquad E^{(p)} \to E$ dual isogeny
  "Verschiebung"
- In our scenario, $\pi : E^{univ} \to Y_{\mathbb{F}_p} \rightsquigarrow \mathrm{Ver}^* : \Omega^1_{E^{univ}/Y_{\mathbb{F}_p}} \to \Omega^1_{E^{univ,(p)}/Y_{\mathbb{F}_p}}$
  induces $\omega = \pi_*(\Omega^1_{E^{univ}/X}) \to \pi_*(\Omega^1_{E^{univ}/X}) \simeq \omega^{(p)} \simeq \omega^p$
  $\Rightarrow \mathrm{Ver}^* \in \mathrm{Hom}(\omega, \omega^p) \simeq \mathrm{Hom}(\mathcal{T}_{\mathbb{F}_p}, \omega^{p-1})$ ⌟

We will use Ha to define a "measure of supersingularity" on pts of $\mathcal{Y}$, which will allow us to define the affinoids $\mathcal{Y}^{\leq v}$ on which the OC forms live.

- Cover $Y_{\mathbb{F}_p}$ by affine opens U_i w/ triv. $s_i : \mathcal{O}_{U_i} \simeq \omega^{p-1}_{\mathbb{F}_p}$
- $U_i \rightsquigarrow \mathcal{U}_i \subset \mathcal{Y}$, $\mathcal{U}_i \subset \mathcal{Y}$
- Choose lifts of each s_i to $\tilde{s}_i : \mathcal{O}_{\mathcal{U}_i} \simeq \omega^{p-1}|_{\mathcal{U}_i}$
- Lift Ha to $\widetilde{\mathrm{Ha}}_i \in H^0(\mathcal{U}_i, \omega^{p-1})$ for each i, & hence a fcn. $f_i \in \tilde{s}_i^{-1}(\widetilde{\mathrm{Ha}}_i) \in \mathcal{O}^+_{\mathcal{Y}}(\mathcal{U}_i)$

Def 2.5: For $v \in [0,1) \cap \mathbb{Q}$, define $\mathcal{Y}^{\leq v}$ to be the union of the aff. subdomains $\mathcal{U}_i^{\leq v} \subset \mathcal{U}_i$ defined by $|f_i| \leq |p|^v$. Namely, $\mathcal{Y}^{\leq v}$ is the locus $v_p(\mathrm{Ha}) \leq v$.



- $\mathcal{Y}^{\le v}$ is indep. of choices made; different triv. $\rightsquigarrow$ lift differs by a unit; different lift $\rightsquigarrow$ $\tilde{f_i}' = \tilde{f_i} + pg$ $\Rightarrow |\tilde{f_i}'| = |\tilde{f_i}|$ if $v<1$
- $\mathcal{Y}^{\le 0} = \mathcal{Y}^{ord}$;
- $\mathcal{Y}^{\le v}$ is obtained from $\mathcal{Y}$ by deleting (the closure of) open discs inside each of the ss residue discs $sp^{-1}(y)$ as y runs over closed pts of $Y_{\mathbb{F}_p}^{ss}$;
- $\mathcal{Y}^{\le v}$ is affinoid.

Def [2.7]: For $v \in [0,1) \cap \mathbb{Q}$, the space of v-OC forms of wt k (and level $\Gamma_1(N)$) is defined to be

$$M_k^{\dagger,v}(N) := H^0(\mathcal{Y}^{\le v}, \omega^k).$$

By construction,

$$\text{Te, Se etn } \quad H^0(Y_{\mathbb{Q}_p}, \omega^k) = H^0(\mathcal{Y}, \omega^k).$$

ue eln

$M_k^{\dagger,v}(N)$ is a f. g. module over $\mathcal{O}(\mathcal{Y}^{\le v}) \rightsquigarrow$ Banach mod. & $\mathbb{Q}_p$-Banach space. (For v suff. small, Coleman: $M_k^{\dagger,v}(N) \simeq \mathcal{O}(\mathcal{Y}^{\le v})$).

III. <u>Up & the canonical subgp.</u>

<u>Goal</u>: equip $M_k^{\dagger,v}(N)$ w/ a compact Hecke opr. U_p.

<u>Thm</u> [2.8] (Katz, Rubin): For $v \le \frac{p}{p+1}$, the section $can: \mathcal{Y}^{ord} \to \mathcal{X}$ extends to $\mathcal{Y}^{\le v}$, and gives an iso. onto its image, denoted by $\mathcal{X}^{m,\le v}$.

On L-pts ($L/\mathbb{Q}_p$ complete), $(E, \eta_N) \longmapsto (E, H_{can}(E), \eta_N)$ where $H_{can}(E)$ is the canonical subgp of $E[p]$. If $C \subset E[p]$ is a rk p loc. free subgp scheme & $C \ne H_{can}(E)$, and $(E,\eta_N) \in \mathcal{Y}^{\le v}$ for $v < \frac{p}{p+1}$, then $(E/C, \bar{\eta}_N) \in \mathcal{Y}^{\le v/p}$ & $H_{can}(E/C) = E[p]/C$.

In particular, $M_k^{\dagger,v}(N) \simeq H^0(\mathcal{X}^{m,\le v}, \omega^k)$, & we can define U_p geometrically as follows:

$$\begin{tikzcd} & Z_{\mathbb{Q}_p} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_{\mathbb{Q}_p} & & X_{\mathbb{Q}_p} \end{tikzcd} \qquad \begin{tikzcd} & (E,H,C) \arrow[dl] \arrow[dr] & \\ (E,H) & & (E/C, E[p]/C) \end{tikzcd}$$

where

- $Z_{\mathbb{Q}_p}$ is a moduli space of quadruples (E, H, C, η_N), the triple (E, H, η_N) gives a pt of $X_{\mathbb{Q}_p}$, & $C \subset E[p]$ is another rk p subgp scheme w/ $C \cap H = \{e\}$.
- $p_2^* \omega^k \simeq p_1^* \omega^k$
- $\exists$ nat'l adjunction map $p_{1,*} p_1^* \omega^k \to \omega^k$, trace on sections

$\rightsquigarrow$

$$H^0(X_{\mathbb{Q}_p}, \omega^k) \xrightarrow{p_2^*} H^0(Z_{\mathbb{Q}_p}, p_2^*\omega^k) \simeq H^0(Z_{\mathbb{Q}_p}, p_1^*\omega^k)$$
$$= H^0(X_{\mathbb{Q}_p}, p_{1,*}p_1^*\omega^k) \xrightarrow{\frac{1}{p}\mathrm{tr}_{p_1}} H^0(X_{\mathbb{Q}_p}, \omega^k).$$

restriction $\rightsquigarrow$

$$\begin{tikzcd} & p^{-1}(\mathcal{X}^{m,\le v}) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{X}^{m,\le v} & & \mathcal{X}^{m,\le v/p} \end{tikzcd}$$

$\rightsquigarrow U_p : H^0(\mathcal{X}^{m,\le v/p}, \omega^k) \longrightarrow H^0(\mathcal{X}^{m,\le v}, \omega^k).$

$\uparrow$ res. $\leftarrow$ compact (similar to $\mathbb{Q}_p\langle pT\rangle \hookrightarrow \mathbb{Q}_p\langle T\rangle$)

$H^0(\mathcal{X}^{m,\le v}, \omega^k)$ $\Downarrow$ U_p compact Prop 2.10

"classicality"

Thm (Coleman): If $h < k-1$, then
$$H^0(X_{\mathbb{Q}_p}, \omega^k)^{\le h} \xrightarrow{\sim} \underline{M_k^+(N)^{\le h}}$$

$F^{+,v}_{k,N}$ = char. power series of U_p is indep. of v.

Rmk: $U_p(\sum_n a_n q^n) = \sum_n a_{np} q^n$, so $\|U_p\| \le 1 \Rightarrow$ slope ≥ 0.

$M_k^{+,v}(N)^Q$ is indep of v for $Q \mid F^+_{k,N}$ $\rightsquigarrow M^+_{k,N}$

§2. The Coleman-Mazur eigencurve

I. Weight space

<u>Notation</u>:

- $\mathcal{W} := \mathrm{Spa}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]], \mathbb{Z}_p[[\mathbb{Z}_p^\times]])_{\mathbb{Q}_p}$

  "represents the functor $R \mapsto \mathrm{Hom}(\mathbb{Z}_p^\times, R^\times)$"

  $\mathrm{Spa}(R, R^+) \to \mathcal{W}$ over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \longleftrightarrow \underbrace{\mathbb{Z}_p^\times \to R^\times}_{\text{" affinoid wt "}}$

More explicitly, $\mathcal{W} = \Delta \times \mathring{\mathbb{D}}(0,1)$, $(p-1)$ copies of open unit discs where Δ = fin. gp. of characters $(\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$.

⌜ $\mathrm{Spa}(R, R^+) \to \mathring{\mathbb{D}}(0,1) = \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]])_{\mathbb{Q}_p} \overset{1-1}{\longleftrightarrow} R^{\circ\circ}$ ⌟

- $k \in \mathbb{Z}$, integral wt. $[k] : \mathbb{Z}_p^\times \to \mathbb{Q}_p^\times$, $z \mapsto z^{k-2}$

2.15

<u>Lem</u>: The integral wts are Zariski-dense in $\mathcal{W}$. Moreover, each int. wt $[k]$ has a basis of aff. nbhds $V \subset \mathcal{W}$ s.t. the int. wts in V are Zariski-dense.

Pf: By $(*)$ above, we can reduce to a statement about pts of the form $(1+p)^k - 1$ in the open unit disc.

Take a seq. $k_m \in \mathbb{Z}$ w/. $k_m \equiv k \ (p-1)p^m \Rightarrow (1+p)^{k_m} - 1$ tends to $(1+p)^k - 1 \Rightarrow \nexists$ an aff. nbhd. of $(1+p)^{k-1}$ which vanish on the seq. $((1+p)^{k_m} - 1)_{m \geq 1}$. $\square$

<u>Rmk</u>*: $M_k^{\dagger,\vee}$ for general k? Coleman constructed these spaces by using the Eisenstein family E_k. Modern approach:

$$\begin{array}{l}
\mathrm{Ig}(p^\infty) \circlearrowleft \mathbb{Z}_p^\times \\
\quad \vdots \\
\mathrm{Ig}(p^n) \circlearrowleft (\mathbb{Z}/p^n\mathbb{Z})^\times \\
\quad \vdots \\
\mathcal{X}^{ord}
\end{array}$$

- $\mathrm{Ig}(p^n) := E[p^n]^{red} \setminus E[p^{n-1}]^{red}$
- $\mathbb{Z}_p^\times \circlearrowright H^0(\mathrm{Ig}(p^\infty), \mathcal{O})$
- $\kappa : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ cts

$\rightsquigarrow M_\kappa^{p\text{-adic}} = H^0(\mathrm{Ig}(p^\infty), \mathcal{O})[\mathbb{Z}_p^\times = \kappa]$

$\rightsquigarrow$ families of OC mod. forms over $\mathrm{Spa}(R, R^+)$.

II. The Coleman-Mazur eigencurve.

Notation:

- $M_{\kappa_U}^{\dagger,U}(N)$: OC forms for $U \subset \mathcal{W}$ aff. open & U suff small.
  $\circlearrowleft U_p$ $\quad \kappa_U : \mathbb{Z}_p^\times \to \mathcal{O}(U)^\times$ aff wt.
- $F_N^\dagger \in \mathcal{O}^+(\mathcal{W})\{\{X\}\}$ char. power series of U_p
- $\mathcal{Z} = V(F_N^\dagger)$, $\mathcal{N}$
- $\mathbb{T}^N := \mathbb{Q}_p$ - alg. gen by $\{U_p, T_\ell, S_\ell : \ell \nmid Np\} \subset H^0(\mathcal{X}^{m,\leq v}, \omega^k)$.
- $\psi : \mathbb{T}^N \to \operatorname{End}(\mathcal{N})$
  $\rightsquigarrow \mathcal{E} = \mathcal{E}_N \xrightarrow{\pi} \mathcal{Z} \to \mathcal{W}$ "classical pts"
- $Z = \{x \in \operatorname{Max}(\mathcal{E}) : w(x) = [k]$ for some $k \in \mathbb{Z}$
  & whose systems of eigenvalues appears in $H^0(X_{\mathbb{Q}_p}, \omega^k)\}$

Goal:

1) The wt map $w : \mathcal{E} \to \mathcal{W}$ is loc. quasi-fin. & flat $\Rightarrow \mathcal{E}$ is equi-dim. of dim 1. ✓ (last time)
2) $\forall$ irr. component $\mathcal{C} \subset \mathcal{E}$, $w(\mathcal{C})$ is the complement of fin. many pts in an irr. component of $\mathcal{W}$
3) $Z \subset \operatorname{Max}(\mathcal{E})$ is Zariski-dense & accumulates at itself.
4) $\mathcal{E}$ is reduced.
5) $\mathcal{E}$ carries a family of Galois (pseudo-) repns : $\exists$ a cts fcn
$$T : G_{\mathbb{Q}} \to \mathcal{O}^+(\mathcal{E})$$
s.t. $\forall x \in Z$, the specialization T_x is the trace of the Galois rep. assoc. to a classical modular form whose Hecke eigenvalues match up w/ x. (next time?)

2.17

Prop: $\forall \mathcal{C} \subset \mathcal{E}$ irr. component, $w(\mathcal{C})$ is the complement of fin. many pts in an irr. component of $\mathcal{W}$. $\quad \chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$

Pf: $\mathcal{E} \to \mathcal{Z} = V(F_N^\dagger)$ is fin. & $\mathcal{E}$ is equidim. of dim. $\dim(\mathcal{Z})$
$\Rightarrow \operatorname{Im}(\mathcal{C}) \subset \mathcal{Z}$ is an irr. component $\mathcal{C}_{\mathcal{Z}}$ $\quad w(\mathcal{C}) \subset \mathcal{W}_\chi$, some irr.

component of $\mathcal{W}$. $\mathcal{W}_x \simeq \mathrm{Spa}(\mathbb{Z}_p[\![t]\!], \mathbb{Z}_p[\![t]\!])_{\mathbb{Q}_p}$. The relevant piece $\mathcal{Z}_x := \mathcal{Z}|_{\mathcal{W}_x}$ is the spectral variety of some $F_x \in \mathbb{Z}_p[\![t]\!]\{\{X\}\}$.

<u>Fact</u>: $\mathcal{C}_{\mathcal{Z}}$ is a spectral var., given by an irr. factor $F_{\mathcal{C}}$ of F_x. Write $F_{\mathcal{C}} = 1 + \sum_{n\geq 1} a_n(t) X^n \in \mathbb{Z}_p[\![t]\!]\{\{X\}\}$. The pts of $\mathcal{W}_x$ that are missing from $w(\mathcal{C}) \leftrightarrow$ common zeros of $\{a_n(t)\}_n \Rightarrow$ fin. by Weierstrass preparation. $\square$.

<u>Thm</u>[2.18]: The subset $\mathcal{Z} \subset \mathrm{Max}(\mathcal{E})$ of classical pts is Zariski-dense and accumulates at itself (i.e., $\forall$ pt z of $\mathcal{Z}$, $\exists$ a basis of aff. nbhds $V \ni z$ s.t. $V \cap \mathcal{Z}$ is Zariski-dense in V).

<u>Pf</u>: Pick $\mathcal{C} \subset \mathcal{E}$ irr. component. Prop$^{2.17} \Rightarrow \exists$ int. wt. $[k_0] \in w(\mathcal{C})$. Choose $x \in w^{-1}([k_0])$. Then $\exists$ open aff. $U \ni x$ in $\mathcal{E}$ w/ $w|_U$ fin. flat & $V = w(U) \subset \mathcal{W}$ conn. aff open.

Now, for $k = k_0 + (p-1)p^M$, $[k] \in V$ if $M \gg 0$. But U_p^{-1} is bdd. on U $\overset{\text{classicality}}{\Longrightarrow}$ elts in $w^{-1}([k]) \cap U$ are classical $\rightsquigarrow$ Zariski-dense family of int. wts in $V \overset{\text{f.f.}}{\Longrightarrow}$ pre-image is Zariski-dense in U & hence in $\mathcal{C}$. $\square$.

<u>Rmk</u>: $\mathcal{Z} \subset \mathrm{Max}(\mathcal{E})$ is NOT dense for the usual top. on $\mathcal{E}$. Moreover, int. wts are NOT —— " —— on $\mathcal{W}$.

On each $\mathcal{W}_x \simeq \mathrm{Spa}(\mathbb{Z}_p[\![T]\!], \mathbb{Z}_p[\![T]\!])_{\mathbb{Q}_p} = D(0,1)$, each int wt lies in one of the open affs $\{|t|_p = p^{-i}\}$ for some $i \in \mathbb{Z}_{>0}$. Hence, e.g. $\{p^{-3/2} \leq |t|_p \leq p^{-4/3}\}$ contains no int wts.

<u>Prop</u>[2.20]: $\mathcal{E}$ is reduced.

<u>Pf</u>: key ingredient: $\mathcal{O}_{\mathcal{E},z}$ is reduced for $z \in \mathcal{Z}$ classical. Then density of classical pts + commutative alg. $\Rightarrow$ result.

For z cls., $\pi(z) \in \mathcal{Z}$ has an open nbhd. $V_{U,h}$ for $U \subset \mathcal{W}$ conn.

aff. open & $h \geq 0$. Let $\tilde{V} = \pi^{-1}(V_{U,h}) \ni z$ open nbhd. w/

$$\mathcal{O}(\tilde{V}) = \mathrm{Im}\left(\mathcal{O}(U) \otimes_{\mathbb{Q}_p} \mathbb{T}^N \to \mathrm{End}_{\mathcal{O}(U)}(M_{k_U}^{+,v}(N)^{\leq h})\right).$$

Consider $[k] \in U$ w/ $h < (k-1)/2$. These $[k]$'s are Zariski-dense in U. By classicality, the image of $\mathcal{O}(\tilde{V})$ in $\mathrm{End}_{\mathbb{Q}_p}(M_{k_U}^{+}(N)^{\leq h})$ is identified w/

$$\mathrm{Im}(\mathbb{T}^N \to \mathrm{End}_{\mathbb{Q}_p}(M_k^{+}(N)^{\leq h}) \to \mathrm{Im}(\mathbb{T}^N \to \underbrace{\mathrm{End}_{\mathbb{Q}_p}(H^0(X_{\mathbb{Q}_p}, \omega^k)^{\leq h})}_{\text{semi-simple}}).$$

the $h < (k-1)/2$ assumption is for up

$\Rightarrow \mathcal{O}(\tilde{V})$ is reduced:

Suppose $t \in \mathcal{O}(\tilde{V})$ w/ $t^n = 0$. Then can identify t w/ an elt in $\mathrm{End}(\underbrace{M_{k_U}^{+,v}(N)}_{:=P})$ & P fin. proj. / $\mathcal{O}(U)$. We know $t \mapsto 0$ in $\mathrm{End}(P \otimes_{\mathcal{O}(U)} m(k))$ for a dense set of max. ideals in $\mathcal{O}(U)$. Extend by 0 to a free mod. w/ P as a direct summand $\rightsquigarrow t \in \mathrm{End}_{\mathcal{O}(U)}$(fin. & free). $\rightsquigarrow$ matrix. Each entry lies in the intersection of a Zariski-dense set of max. ideals. $\mathcal{O}(U)$ is reduced $\Rightarrow$ each entry is $0 \Rightarrow t = 0$.

$\square$.

§3. Open questions

* Fin. of irr. components?
* Properness of the wt map (Diao–Liu '16)
* Ghost conjecture

$f = \sum a_n q^n \in S_k(\Gamma_0(pN))$ normalized eigenform, then $v(a_p) \in [0, k-1]$

Which values can $v(a_p)$ take? How do they distribute?

Conj (Gouvêa): Let $\{\alpha_1, \cdots, \alpha_t\}$ be the multi-set of slopes in $S_k(\Gamma_0(pN))$ & $\mu_k = \frac{1}{k-1}$ $\curvearrowright$ $\subset [0,1]$. Then

$$\lim_{k\to\infty} \mu_k = \frac{p-1}{p+1}\delta_{\frac{1}{2}} + \frac{1}{p+1}\delta_{[0,\frac{1}{p+1}]} + \frac{1}{p+1}\delta_{[\frac{p}{p+1},1]}$$

↳ p-new ↳ p-old

Conj (Spectral Halo): Near the boundary of the wt space, $\mathcal{E}$ = countably infinitely many disjoint union of annulus, & slopes of pts in each annulus is proportional to the p-adic valuation of the wt.

Conj (Ghost): Under certain conditions, slope of pts on $\mathcal{E}$ can be written via combinatorics (the ghost series).
↳ "Buzzard"; "Robert-Pollack"

$\Rightarrow \begin{cases} \text{Spectral Halo conj.} \\ \text{fin. of irr. components} \\ \text{Gouvêa} \end{cases}$

Other related questions:

* Local constancy of slope (see e.g. [LHV] §2 of ch. 6)
* Gouvêa's $\lfloor \frac{k-1}{p+1} \rfloor$ conj.
* ...

Significant progress has been done. see [LTXZ23]!