Talk 6. The eigenvariety machine II: construction & first properties

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p-adic family / deformation

Plan for today:

1) Spectral variety
2) the eigenvariety machine
3) what's next?

- p-adically interpolate systems of Hecke eigenvalues
- parameter space for systems of Hecke eigenvalues attached to p-adic forms

pts $\longleftrightarrow$ Hecke eigensystems

§1. Overview

Construction of eigenvarieties can be viewed as a 3-step process.

Input data
- wt. space
- spaces of p-adic auto. forms

3)

$\Downarrow$

$\mathcal{Z} \to \mathcal{W}$ spectral variety
- parametrizes non-zero e-values of one special Hecke opr.

$\Downarrow$

$\mathcal{E} \to \mathcal{Z} \to \mathcal{W}$ eigenvariety
- fin. over $\mathcal{Z}$
- parametrizes systems of e-values

Adic v.s. rigid analytic

- Hida: $\mathrm{Spf}\, \mathbb{Z}_p[[T]]$ Integral model
- Coleman – Mazur: $\mathcal{W}^{rig} := \mathrm{Spf}\, \mathbb{Z}_p[[T]]_{\mathbb{Q}_p}$

$$= \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]]) \times_{\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$$

open unit disc   analytic locus

- Us: $\mathcal{W} := \mathrm{Spa}(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]])^{an} = \mathfrak{D} + x_{\mathbb{F}_p[[T]]}$

+ quasi-cpt. + bridge to char p.

§2. Spectral varieties

Notation:

- (A, A^+) Tate-Huber pair w/ pseudo-uniformizer $\varpi \in A$ strongly Noetherian (e.g. $K\langle T_1, \cdots, T_n\rangle / I$) $\Rightarrow$ sheafy
- $W = \mathrm{Spa}(A, A^+)$, $\mathbb{A}^1_W = \bigcup_{n\geq 1} \mathbb{D}_n$ ← disc of radius ϖ^{-n}
- $\mathbb{D}_\lambda = \{x \in \mathbb{A}^1_W : |\varpi^\ell X^m(x)| \leq 1\} \subset \mathbb{A}^1_W$, $\forall \lambda = \ell/m \in \mathbb{Q}$ w/ $\ell \in \mathbb{Z}$, $m \in \mathbb{N}$.
- $F = \sum_{n\geq 0} a_n X^n \in A\{\{X\}\} = \mathcal{O}_{\mathbb{A}^1_W}$ Fredholm series, i.e. $a_0 = 1$ (e.g. char. power series of a cpt opr)
  $\rightsquigarrow$ coherent $\mathcal{O}_{\mathbb{A}^1_W}$-ideal $\mathcal{I}_F$: $F \in A\langle \varpi^n X\rangle$, $\forall n \geq 1 \Rightarrow$ can take $\mathcal{I}_F|_{\mathbb{D}_n} := (\tilde{F})$

Def: The spectral variety (Fredholm hypersurface) of F is def. as the closed adic subspace

$$V(F) := V(\mathcal{I}_F) \subset \mathbb{A}^1_W.$$

More explicitly, on $\mathbb{D}_n$ we set

$$V(F) \cap \mathbb{D}_n = \mathrm{Spa}(B_n, B_n^+),$$

where $B_n := A\langle \varpi^n X\rangle / (F)$ & $B_n^+ := (A\langle \varpi^n X\rangle / (F))^+$.

Rmk: The formation of the spectral variety behaves well w.r.t. base change.

$$h : (A, A^+) \to (B, B^+) \rightsquigarrow \overset{\mathrm{Spa}(B,B^+)}{\overset{\shortparallel}{Y}} \to W$$

$\Rightarrow$

$$V(F) \times_W Y \simeq V(h(F)) \subset \mathbb{A}^1_Y.$$

Now, we have a nat'l structural map

$w : V(F) \to \mathrm{Spa}(A, A^+)$

WTS: w is

1) flat; ⚠ skip, error in the Heidelberg notes
2) loc. quasi-fin.; ⇜ $F \in K\{\{X\}\}$ has fin. many zeros on $\mathcal{D}_{\lambda n}$ "Weierstrass theory Newton polygons"
3) partially proper.

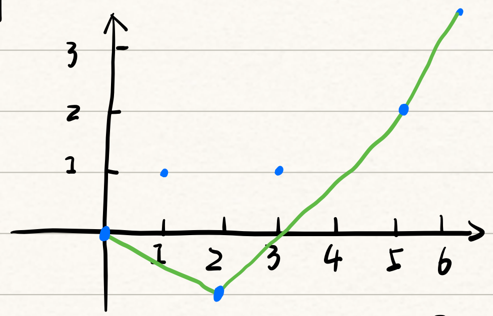
Newton polygons

- Toy case: polynomials

  K non-Archimedean, $f(x) = 1 + a_1 x + \cdots + a_n x^n$

  ⇝ $(0,0), (1, v_p(a_1)), \cdots, (n, v_p(a_n))$ & take lower convex hull

  e.g. $f(x) = 1 + 5x + \frac{1}{5}x^2 + 35x^3 + 25x^5 + 625x^6$



  - slope $-1/2,\ 1,\ 2$
  - length $2,\ 3,\ 1$
  - breaks $0,\ 2,\ 5,\ 6$

  If the Newton polygon of $f(x)$ has its first break at (i, m_i), then $\exists\, g(x), h(x) \in K[x]$ s.t.

  1) $f(x) = g(x)h(x)$; ← all zeros are of abs. val. p^m
  2) $g(x)$ has deg i & is pure of slope m;
  3) $h(x)$ has no zeros in $\mathbb{B}(p^m, 0)$.

  Repeat the process for each break ⇝ for f w/ slopes m_k, $1 \le k \le r$ & lengths i_k, $1 \le k \le r$, f has exactly i_k roots of val. p^{m_k}.

- General case: power series

  e.g. $f(x) = 1 + px + px^2 + px^3 + \cdots$? need to amend rules (see Gouvêa)

  - sup of slope ⟷ radius of conv.
  - the result above still holds: if f conv. on the closed ball

of radius p^m, & slopes $m_k \le m$, $1 \le k \le r$, lengths i_k, $1 \le k \le r$, then for each k, f has i_k zeros of val. $-m_k$ and no other zeros in $\bar{\mathbb{B}}(p^m, 0)$. More precisely,

3.7 **Lem**: Let $s_1, \cdots, s_k$ be the first k slopes of the Newton polygon ($=: \mathcal{N}(F)$) of F. Let n_0 be the x-coord. of the right endpt. of the k-th segment of the Newton polygon. Then $\exists$ $P \in K[X]$ of deg n_0 & $S \in K[\![X]\!]$ s.t.

1) $F = PS$;
2) S is entire;
3) the Newton polygon of P is equal to the portion of the Newton polygon contained in the region $0 \le x \le x_0$;
4) S has slope $> s_k$.

3.6 **Lem**: $f: X \to Y$ morphism of analytic adic spaces of fin type. Then f is quasi-fin. $\Leftrightarrow$ $\forall$ rk 1 pt. $y \in Y$, the mor. $X \times_Y \mathrm{Spa}(k(y)) \to \mathrm{Spa}(k(y))$ is fin.

Pf: See [Hub98], Lem 1.5.2. □.

3.8 **Lem**: Suppose $A = K$ is a non-Arch. field. Then $\exists$ strictly increasing seq. $(\lambda_n) \in \mathbb{Q}_{>0}^{\mathbb{N}}$ w/ $\lambda_n \to \infty$, s.t. $V(F) \cap \mathbb{D}_{\lambda_n}$ is fin. over $\mathrm{Spa}(K, K^+)$.

Pf: WLOG, F is not a polynomial. $\Rightarrow$ $\mathcal{N}(F)$ has ∞-many segments & ∞-many slopes, denoted $\{s_n\}_{n \in \mathbb{N}}$. For any $n \in \mathbb{N}$, pick $\lambda_n = \ell/m \in \mathbb{Q}$ s.t. $s_n < \lambda_n < s_{n+1}$. Claim: $V(F) \cap \mathbb{D}_{\lambda_n}$ is fin. / $\mathrm{Spa}(K, K^+)$.

* WLOG, $\exists$ $\beta \in K$, s.t. $v(\beta) = \lambda_n$ (base change to a fin.

extn. if necessary) $\Rightarrow D_n = \mathrm{Spa}(K\langle\beta X\rangle, K^+\langle\beta X\rangle)$.

* $F = P \cdot S$ as above. so S Fredholm w/ slope $> \lambda_n$.

In $\bar{K}$, $P(X) = \prod(1-\alpha_i X)$ w/ $v(\alpha_i) < \lambda_n$, $\forall i$

$$= \prod \beta^{-1}\alpha_i \underbrace{\prod(\alpha_i^{-1}\beta - \beta X)}_{:= Q(X) \in K^{oo}\langle\beta X\rangle}$$

Now, note that $S(X)$ is a unit in $K\langle\beta X\rangle$. we see

$$K\langle\beta X\rangle/(F(X)) = K\langle\beta X\rangle/(Q(X))$$

$$= K\langle T\rangle/(Q'(T)) \quad (T=\beta X)$$

$\uparrow$ fin.

$$(K\langle\beta X\rangle/(F(X)))^+ = K^+\langle T\rangle/(Q'(T))$$

$\square$.

Thm[3.9]: The morphism $w: V(F) \to W = \mathrm{Spa}(A, A^+)$ is flat, loc. quasi-fin. & partially proper. $\Leftrightarrow$ $\forall x \in V(F)$, $\exists$ open nbhd U of x in $V(F)$ s.t. $\overline{\{x\}} \subset U$, & an open nbhd V of $w(x)$ in W s.t. $\overline{\{w(x)\}} \subset V$ and $w(U) \subset V$, and $w|_U = U \to V$ is fin. & flat. ← Lem. 3.5

$\Uparrow$ $\Leftrightarrow$ loc. quasi-fin. + partially proper

Y, Z loc. strongly Noetherian analytic, $f: Y \to Z$ fin. flat $\Rightarrow f_* \mathcal{O}_Y$ is a fin. loc. free $\mathcal{O}_Z$-mod. We say f has const. deg. if $f_*\mathcal{O}_Y$ has rk d over $\mathcal{O}_Z$. If Z is conn., then every fin. flat map is of const. deg.

Cor[3.12]: $\mathrm{Cov}(V(F)) := \{$open affinoid $U \subset V(F)$ s.t. $w(U) \subset W$ is open affinoid & $w|_U : U \to w(U)$ is fin. of const. deg.$\}$ Then $\mathrm{Cov}(V(F))$ is an open cover of $V(F)$.

☆ "an adm. cover in [Buz07]"

Rmk: The reason we want this cor. is that opens in $\mathrm{Cov}(V(F))$

are the affinoids over which one can construct a Hecke alg & hence an eigenvariety.

3.14

Prop: $x \in V(F)$, $U = \mathrm{Spa}(C, C^+) \subset V(F)$ & $V = \mathrm{Spa}(B, B^+) \subset W$ be affinoid opens as above. Assume that U is of const. deg. d over V. Then F has a factorization in $B\{\{X\}\}$ of the form $F = QS$, where $Q = 1 + b_1 X + \cdots + b_d X^d \in B[X]$, $b_d \in B^\times$ & S is a Fredholm series w/ coeffs in B, $(S, Q) = 1$ & $C = B[X]/(Q)$.

Conversely, for such a factorization $F = QS$ in $B\{\{X\}\}$,

$$U = \mathrm{Spa}(B[X]/(Q), (B[X]/(Q))^+)$$

is nat'lly an elt of $\mathrm{Cov}(V(F))$.

§3. The eigenvariety machine

I. Eigenvarieties over an affine base

Notation:

- (A, A^+), A $\mathcal{O}_K$-alg. ; $W = \mathrm{Spa}(A, A^+)$
- M Banach A-mod.
- $\mathbb{T}$ comm. $\mathcal{O}_K$-alg. w/ $\mathbb{T} \to \mathrm{End}_A(M)$, we often identify $t \in \mathbb{T}$ w/ its image in $\mathrm{End}_A(M)$.
- ϕ : fixed elt in $\mathbb{T}$, $\phi : M \to M$ cpt w/ $F(X) = 1 + \sum_{n \geq 1} c_n X^n = \det(1 - X\phi)$
- $Z := V(F) \subset \mathbb{A}^1_W$ spectral var. of F

$((A, A^+), M, \mathbb{T}, \phi)$ "spectral datum" $\overset{\text{Goal}}{\rightsquigarrow} \mathcal{E}$

e.g. M f.g. proj. of rk d, ϕ inv., $\mathbb{T}$ A-alg., $P = \det(1 - X\phi)$

$Z := \mathrm{Spa}(A[X]/P(X), (A[X]/(P(X)))^+)$.

Let $\mathbb{T}(Z) :=$ image of $\mathbb{T}$ in $\mathrm{End}_A(M)$, then $\mathbb{T}(Z)$ fin. By Hamilton-Cayley $\phi^{-1} \in \mathbb{T}(Z)$ & $\exists$ nat'l map

$$A[X]/(P(X)) \to \mathbb{T}(Z)$$

$$x \longmapsto \phi^{-1}$$

$\rightsquigarrow \mathcal{E} := \mathrm{Spa}(\mathbb{T}(Z), \mathbb{T}(Z)^+) \longrightarrow Z \longrightarrow W$.

Geometry of $\mathcal{E}$ depends on the "Hecke oprs": e.g. $A = K\langle T_1, T_2\rangle$, $M = A^2$, $\mathbb{T} = A[\phi, t]$ w/ $\phi = \begin{pmatrix} 1 & T_1 \\ & 1 \end{pmatrix}$ & $t = \begin{pmatrix} 0 & T_2 \\ 0 & 0 \end{pmatrix}$. $\Rightarrow \det(1 - X\phi) = (1-X)^2$, so Z is non-red. $\mathbb{T}(Z) \simeq A \oplus I\varepsilon$ w/ $I = (T_1, T_2)$ & $\varepsilon^2 = 0$. $\Rightarrow \mathcal{E} \to Z$, $\mathcal{E} \to W$ not flat & $\mathcal{E}$ is non-red. ⌟

$\phi \mapsto 1 + \varepsilon T_1$

$t \mapsto \varepsilon T_2$

<u>Construction</u>:

- $U \in \mathrm{Cov}(Z)$, so $w|_U : U \xrightarrow{ff} V = w(U) = \mathrm{Spa}(B, B^+) \subset W$
- $M_B := M \hat{\otimes}_A B$
- $\forall\, t \in \mathbb{T}$, $t_U := t \otimes 1$ cpt, $F_U(X) = \mathrm{Im}(F) \in B\{\{X\}\}$
- $\mathcal{O}_Z(U) = B[X]/(Q_U)$ w/ $Q_U = 1 + b_1 X + \cdots + b_d X^d$ w/ $b_d \in B^\times$, & $F_U = Q_U S_U$ w/ $(S_U, Q_U) = 1$.
- $M_B = N \oplus M(Q)$ w/ $N = \mathrm{Ker}(Q^*(\phi_U))$ proj. of rk d / B; t-inv. for all $t \in \mathbb{T}$.
- & fin.
  $\mathbb{T}_U :=$ B-subalg. of $\mathrm{End}_B(N)$ gen. by $t \in \mathbb{T}$. $\Rightarrow \mathbb{T}_U$ Tate-Huber.
- $\mathcal{E}_U := \mathrm{Spa}(\mathbb{T}_U, \mathbb{T}_U^+)$.

By construction,

$$B[Y]/(Q^*(Y)) \longrightarrow \mathbb{T}_U$$

$$Y \longmapsto \phi \quad \rightsquigarrow \mathcal{E}_U \xrightarrow{\text{fin.}} U$$

is

$$B[X]/(Q(X))$$

$$Y \mapsto X^{-1}$$

Now, WTS: these local pieces glue when U ranges through elts of $\mathrm{Cov}(Z)$.

* $U \mapsto \mathrm{Ker}(Q_U^*(\phi_U))$ defines a coherent sheaf $\mathcal{N}$ of $\mathcal{O}_Z$-mod. on Z, i.e., for $U_1 \subset U_2$,

$$\mathrm{Ker}(Q_{U_2}^*(\varphi_{U_2})) \otimes_{\mathcal{O}_{U_2}} \mathcal{O}_{U_1} \simeq \mathrm{Ker}(Q_{U_1}^*(\varphi_{U_1})).$$

⌜ Put $U_3 = w|_{U_2}^{-1}(w(U_1))$. $\rightsquigarrow U_1 \subset U_3 \subset U_2$. ISTS:

- $U_1 \subset U_3$, $w(U_1) = w(U_3)$. $\rightsquigarrow Q_{U_1} | Q_{U_3}$

  Fact: (Prop 5.4) $F = Q_1 S_1 = Q_2 S_2$ w/ Q_i mult., S_i Fred., $(S_i, Q_i) = 1$, $Q_1 | Q_2 \implies$

  $$\mathrm{Ker}(Q_2^*(\varphi)) \otimes_{A[X]/(Q_2)} A[X]/(Q_1) \simeq \mathrm{Ker}(Q_1^*(\varphi)).$$

- $U_3 \subset U_2$, $U_3 = U_2 \times_{w(U_2)} w(U_1)$ + base change for spectral var. ⌟

* $\forall U \in \mathrm{Cov}(Z)$, $\mathbb{T} \to \mathrm{End}_{\mathcal{O}_Z}(\mathcal{N})(U)$, $\mathbb{T}(Z) :=$ sub-presheaf of $\mathrm{End}_{\mathcal{O}_Z}(\mathcal{N})$ gen. over $\mathcal{O}_Z$ by the image of $\mathbb{T}$, which is a sheaf by flatness of rat'l localization.

$\implies \mathcal{E} := \mathrm{Spa}(\mathbb{T}(Z))$. "eigenvariety"

By construction, $\mathcal{E} \to Z$ fin. $\implies \mathcal{E}$ is quasi-sep. & loc. quasi-fin. & partially proper over W.

Lem 5.7: $(A, A^+) \to (A', A'^+) \rightsquigarrow Z' = Z \times_{\mathrm{Spa}(A,A^+)} \mathrm{Spa}(A', A'^+)$, $\mathcal{E}' \to \mathcal{E}$ over Z & $\mathcal{E}' \to \mathcal{E} \times_Z Z'$ induces $(\mathcal{E}')^{red} \simeq (-)^{red}$. If $A \to A'$ is flat, then $\mathcal{E}' \xrightarrow{\sim} \mathcal{E} \times_Z Z'$.

II Eigenvariety data

- W loc. Noe. analytic adic space / $\mathcal{O}_K$

  $\cup$

  $\mathrm{Spa}(A, A^+)$ w/ M, $\mathbb{T}$, φ

  ⚠ dep. on A ↑ indep. of A

  $\rightsquigarrow$ glue ??

  * Method 1: the M's are "linked" [Buz 07]

  * Method 2: Fredholm series & spectral var. indep. of A

    $\rightsquigarrow$ "global" $Z \to W$, & $\mathcal{N}$ glues

<u>Def</u>: An eigenvariety datum is a tuple $\mathcal{D} = (\mathcal{Z}, \mathcal{N}, \mathbb{T}, \psi)$, where $\mathcal{Z}$ is a loc. Noe. analytic adic space / $\mathcal{O}_K$, $\mathcal{N}$ is a coh. $\mathcal{O}_{\mathcal{Z}}$-mod., $\mathbb{T}$ a comm. $\mathcal{O}_K$-alg. and $\psi: \mathbb{T} \to \mathrm{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{N})$ an $\mathcal{O}_K$-alg. hom.

<u>Prop</u> 5.9: Given $\mathcal{D}$, $\exists$ loc. Noe analytic adic space $\mathcal{E} = \mathcal{E}(\mathcal{D})$ over $\mathcal{O}_K$ & $\pi: \mathcal{E} \to \mathcal{Z}$ fin., an $\mathcal{O}_K$-alg. hom $\varphi_{\mathcal{E}}: \mathbb{T} \to \mathcal{O}(\mathcal{E})$, and a faithful coherent $\mathcal{O}_{\mathcal{E}}$-mod. $\mathcal{H}$. $\exists$ can. iso.

$$\pi_* \mathcal{H} \cong \mathcal{N}$$

which is compatible w/ actions of $\mathbb{T}$ on both sides.

$\mathcal{E}$ is characterized by the following local description: For $\mathcal{U} \subset \mathcal{Z}$ aff. open, $\mathcal{E}_{\mathcal{U}} = \mathrm{Spa}(\mathbb{T}_{\mathcal{U}}, \mathbb{T}_{\mathcal{U}}^+)$, where $\mathbb{T}_{\mathcal{U}}$ is the $\mathcal{O}_{\mathcal{Z}}(\mathcal{U})$-subalg. of $\mathrm{End}_{\mathcal{O}_{\mathcal{Z}}(\mathcal{U})}(\mathcal{N}(\mathcal{U}))$ gen by the image of $\mathbb{T}$, $\mathbb{T}_{\mathcal{U}}^+$ the int. cl. of $\mathcal{O}_{\mathcal{Z}}^+(\mathcal{U})$ in $\mathbb{T}_{\mathcal{U}}$. As $\mathcal{N}(\mathcal{U})$ is can. a $\mathbb{T}_{\mathcal{U}}$-mod., this gives $\mathcal{H}$ over $\mathcal{E}_{\mathcal{U}}$.

<u>Pf</u>: For $\mathcal{U} \subset \mathcal{Z}$ aff. open, $\mathbb{T}_{\mathcal{U}}$ comm. & fin. over $\mathcal{O}_{\mathcal{Z}}(\mathcal{U}) \Rightarrow$ $(\mathbb{T}_{\mathcal{U}}, \mathbb{T}_{\mathcal{U}}^+)$ is a Noe. Tate-Huber pair. The space $\mathcal{E}_{\mathcal{U}}$ comes w/ a can. fin. map $\mathcal{E}_{\mathcal{U}} \to \mathcal{U}$ & $\mathcal{N}(\mathcal{U})$ is a f.g. $\mathbb{T}_{\mathcal{U}}$-mod. By base change + flatness of rational localization for aff. Noe. analytic adic spaces, these constructions glue together & satisfy the statements. $\square$.

III. <u>Points of eigenvarieties w/ "pseudorigid" base</u>

<u>Notation</u>

+ well-behaved set of max. pts $\leftrightarrow$ max ideals of rings underlying the affinoids

- $\mathcal{W}$ "pseudorigid", $\mathcal{E} \to \mathcal{Z} \to \mathcal{W}$, with composite $\kappa: \mathcal{E} \to \mathcal{W}$

  $\mathcal{U}$ open cover
- $W = \mathrm{Spa}(A, A^\circ)$ & $((A, A^\circ), M_A, \mathbb{T}, \varphi)$; $\mathcal{Z}_W = \mathcal{Z} \times_{\mathcal{W}} W$

- $M_w := M_A \hat{\otimes}_A L$ for $w \in \mathrm{Max}(A)$ w/ residue field $L = k(w)$.
- $\mathrm{Max}(\mathcal{E}) := \{ x \in \mathcal{E} : \exists$ aff. open pseudorigid $\mathcal{U} = \mathrm{Spa}(A) \ni x$ s.t. x corresp. a max ideal of $A \}$

Prop. 5.12

<u>Claim</u>: For L'/L fin. extn,

$$\{ L'\text{-pts } x \in \mathrm{Max}(\mathcal{E}) \text{ s.t. } \kappa(x) = w \} \qquad x$$

$$\updownarrow 1\text{-}1 \qquad\qquad \downarrow$$

$$\left\{ \begin{array}{c} \phi\text{-fin. systems of e-values of } \mathbb{T} \\ \text{appearing in } M_w \otimes_L L' \end{array} \right\} \qquad \begin{array}{l} \lambda_x : \mathbb{T} \to L' \\ t \mapsto \varphi_{\mathcal{E}}(t)(x) \end{array}$$

- $A = L$ a field, $\mathbb{T} \curvearrowright M_L$, v.s. $/L$; so $m \in M_L$ common e-vector $\rightsquigarrow \lambda : \mathbb{T} \to L$ hom.; conversely,
- $\lambda : \mathbb{T} \to L$ appear in M_L if $M_L[\lambda] \neq 0$.
  "ϕ-finite" if $\lambda(\phi) \neq 0$

Idea: λ appears in $(M_w \otimes_L L')$ is ϕ-fin. $\Leftrightarrow$ it appears in $\underbrace{(\mathcal{N}(\mathcal{U}) \otimes_A L)}_{:= N_w} \otimes L'$ for some $\mathcal{U} \in \mathrm{Cov}(\mathcal{Z})$ w/ $w \in \mathrm{Im}(\mathcal{U}) = V = \mathrm{Spa}(A)$. On the other hand, an L'-pt. $x \in \mathrm{Max}(\mathcal{E}_\mathcal{U})$ gives a hom. $\mathbb{T}_\mathcal{U} \to L'$. $\Rightarrow$ ISTS:

$$\{ \text{systems of e-values in } N_w \otimes_L L' \} \overset{1\text{-}1}{\longleftrightarrow} \mathrm{Spec}\, \mathbb{T}_\mathcal{U} \otimes L\, (L').$$

Now, N_w is a fin. dim. vector space $/L$, w/ an action of $T_L \subset \mathrm{End}_L(N_L)$ gen. by image of $\mathbb{T} \otimes_{\mathcal{O}_K} L$.

$*$ T_L is fin. $/L$ $\Rightarrow$ Artinian & semi-local

[Bel] §2.5.1 $\Rightarrow \forall \underset{\text{max.}}{m} \subset T_L$, $N_L[m] \neq 0$ (assuming $T_L \curvearrowright N_w$ faithfully?)

[Bel] Cor 2.5.10
$*$ $\lambda : \mathbb{T} \to L$ appears in $N_w \Leftrightarrow$ it factors through T_e

$*$ $\forall L'/L$,

$$\mathrm{Spec}\, T_L\, (L') \overset{1\text{-}1}{\longleftrightarrow} \{ \text{systems of e-values in } N_w \otimes_L L' \}$$

$$\updownarrow 1\text{-}1$$

$$\mathrm{Spec}\, \mathbb{T}_\mathcal{U} \otimes_A L\, (L')$$

$\overset{1\text{-}1}{\longleftrightarrow}$

Lem 5.6

since the kernel of the map $\mathbb{T}_\mathcal{U} \otimes_A L \to T_L$ is nilpotent.

§4. What's next?

We are going to work through various examples of eigenvarieties constructed via different approaches:

- The Coleman-Mazur eigencurve ← "coherent coh of Shimura vars"
  - geometric theory of mod. forms (Talk 7, Haoran)
  - Galois theoretic construction (Talk 8, Yicheng)

  ⇝ AIP, Boxer-Pilloni, Kisin, ...
- Eigencurves for definite quaternion algs (Talk 9, Benchao)
  * inspired by Stevens' overconv. mod. symbols
  * Jacquet-Langlands corresp. & comparison (Thm 5.17)
  * alg. mod. forms, loc. analytic repns, ...

  } warm-up for ← GL_2 / def. quat.
- Stevens' overconv. mod symbols (Talks 10-11, Pengcheng, Zhenghang)
  * classical & overconv. mod symbols (local systems, repns of $\pi_1(X)$, *flat vector bundles, ...)
  * distribution modules / cohomology, comparison ← "sing. coh. of loc. sym spaces"
  * construction of p-adic L-fcns ← G red., |cpt (mod center)|

  ⇝ Ash-Stevens, Urban, Hansen, ...
- Emerton's approach (Remaining talks, Benchao, Dealing, Yicheng)
  * completed cohomology
  * loc. analytic Jacquet functor ← G red.
  * repns of p-adic groups gps