Adic Spaces – some topics

Warning, I didn't spend a lot of time typing so some stuff might be wrong.

1 Coherent Sheaves

1.1 small amount of history

Historically the theory of Tate's rigid analyitic varieties was developed first, and they were defined as G-topologized spaces $(X, \mathcal{O}_X)$ locally looking like a Tate-algebra. However then the underlying space X carried only a Grothendieck topology, and was no actual topological space. This resulted in bad sheaf theoretic behaviour (for example a nonzero sheaf that has zero stalks everywhere). In the theory of adic spaces we have an actual underlying topological space. This part completely follows the notes [Eug24].

Assume that (A, A^+) is a noetherian Huber pair. We want to associate a sheaf to A-modules, ideally taking values in complete topological rings.

Proposition 1.1. Let (A, A^+) be a noetherian Huber pair and let M be an A-module. Then the presheaf $\tilde{M}(U) = M \otimes_A \mathcal{O}_X(U)$ on $X = \mathrm{Spa}(A, A^+)$ is a sheaf, and $H^i(X, \tilde{M}) = 0$ for all $i \geq 1$. Moreover, the assignment $M \mapsto \tilde{M}$ defines an exact functor from A-modules to $\mathcal{O}_X$-modules.

If we focus on coherent sheaves, i.e. those coming from finitely generated A-modules, we obtain a valid definition for general locally noetherian adic spaces, valued in complete topological rings (with the following topology).

Proposition 1.2. Assume that (A, A^+) is a noetherian Huber pair and that M is a finitely generated A-module. If $A^n \longrightarrow M$ is a surjection, we equip M with the quotient topology coming from this surjection, where A^n has the product topology. This topology is called the canonical topology and is complete and independent of the choice of surjection from a finitely generated free module.

The following results implies that fact that a sheaf is coming from a fin. gen. module does not depend on the cover.

Proposition 1.3. Let (A, A^+) be a noetherian Huber pair with $X = \mathrm{Spa}(A, A^+)$ and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. Assume that there is a cover (U_i) of rational subsets and finitely generated $\mathcal{O}_X(U_i)$-modules M_i such that $\mathcal{F}|_{U_i} = \tilde{M}_i$ for all i. Then there is a finitely generated A-module M such that $\mathcal{F} = \tilde{M}$.

So now we can define general coherent sheaves.

Definition 1.4. Let X be a locally noetherian adic space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called coherent if there is a cover (U_i) of X by affinoid opens $U_i = \mathrm{Spa}(A_i, A_i^+)$ and finitely generated A_i-modules M_i such that $\mathcal{F}|_{U_i} = \tilde{M}_i$ for all i.

2 Finiteness properties of morphisms

2.1 Morphisms of finite type

Also completely follows the notes [Eug24] chapter 2, part 7.

We would like to define what *finite type* means. We cannot proceed as for schemes since if we were to just base it on rings maps of finite type, the Tate algebra $k\langle T\rangle$ is not of finite type over k. Instead, we have to take a big detour.

Definition 2.1. A homomorphism $R \longrightarrow S$ of complete Huber rings is a quotient map if it is surjective, continuous, and open. A homomorphism $(R, R^+) \longrightarrow (S, S^+)$ of Huber pairs is a quotient map if $R \longrightarrow S$ is a quotient map and S^+ is the relative integral closure of the image of R^+ in S.

Definition 2.2. A homomorphism $R \longrightarrow S$ of Huber rings is *strictly of topologically finite type* if there is a quotient map $R\langle T_1, \ldots, T_n\rangle \longrightarrow \hat{S}$ of $\hat{R}$-algebras.

Example 2.3. Now all Tate algebras $k\langle T_1, \ldots, T_n\rangle$ and quotients of it are strictly of topologically finite type over k.

Example 2.4. Sadly, $\mathbb{Q}_p$ is not strictly of topologically finite type over $\mathbb{Z}_p$. One might think that $\mathbb{Z}_p\langle T\rangle \longrightarrow \mathbb{Q}_p$, sending T to $1/p$ works, but in fact $\frac{1}{1-pT}$ is an element in $\mathbb{Z}_p\langle T\rangle$ but this cannot map to anything in $\mathbb{Q}_p$.

This notion only includes polydisks and adic spaces cut out by finitely many equations from polydisks.

Let R be a Huber ring, let $M = (M_1, \ldots, M_n)$ be a tuple of finite subsets of R. Set

$$M_i^r R := \langle m_1 \ldots m_r a \mid m_j \in M_i,\ a \in R\rangle$$

is open. For $i \in \mathbb{N}^n$ and U a neighbourhood of 0 in R set $M^i U := M_1^{i_1} \cdots M_n^{i_n} U$. We call a tuple $M = (M_1, \ldots, M_n)$ *voluminous* if $M^i R$ is open for all $i \in \mathbb{N}^n$. For a voluminous tuple M, write T for $(T_1, \ldots, T_n)$ and

$$R\langle T\rangle_M = \{\sum_{i\in\mathbb{N}^n} a_i T^i \mid \text{for all } U \text{ nbd of 0 in } R\text{: } a_i \in M^i U \text{ for almost all } i\}$$

Remark 2.5. This definition might seem odd, but let us consider the case where k is a non-archimedean field with norm $||\cdot||$, $M = \{\alpha\}$, $r = ||\alpha||$. Then

$$k\langle T\rangle_M = \{\sum_i a_i T^i \mid ||a_i|| r^i \to 0\}$$

i.e. the power series such that the Gauss norm converges.

Example 2.6. We have a surjection

$$\begin{aligned} \mathbb{Z}_p\langle T\rangle_p &\longrightarrow \mathbb{Q}_p \\ T &\longrightarrow 1/p \end{aligned}$$

Indeed, $\frac{1}{1-p}T$ is not in $\mathbb{Z}_p\langle T\rangle_p$.

Definition 2.7. Let $R \longrightarrow S$ be a homomorphism of Huber rings. Then S is of topologically finite type over R if there is a voluminous tuple $M = (M_1, \dots, M_n)$ of finite subsets of R and a quotient map of R-algebras:

$$R\langle T\rangle_M \twoheadrightarrow S.$$

Proposition 2.8. Any homomorphism $R \longrightarrow S$ of Tate rings of topologically finite type is strictly of topologically finite type.

Definition 2.9. Let $\varphi : (R, R^+) \longrightarrow (S, S^+)$ be a homomorphism of complete Huber pairs. Then φ is

1. weakly of topologically finite type if $R \longrightarrow S$ is of topologically finite type,
2. of topologically finite type if there is a quotient map

$$(R\langle T\rangle_M, R\langle T\rangle_M^+) \twoheadrightarrow (S, S^+)$$

of Huber pairs over (R, R^+) for some voluminous tuple M.

Example 2.10. 1. Examples of topologically finite Huber pairs are:

(a) $(R, R^+) \longrightarrow (R\langle T\rangle, R^+\langle T\rangle)$

(b) for a rational open $U \subset R$, $(R, R^+) \longrightarrow (R_U, R_U^+)$ (recall $R_U = R[1/g]$, $R_U^+ = R[\frac{f_1}{g}, \dots, \frac{f_n}{g}]^N$

2. Examples of topologically finite Huber pairs are:

(a) $(R, R^+) \longrightarrow (R\langle T_1, \dots\rangle, R^+\langle T_1, \dots\rangle)$

(b) $(R, R^+) \longrightarrow (R[[T]], R^+[[T]])$

(c) $(R, R^+) \longrightarrow (R_{\mathrm{disc}}, R^+_{\mathrm{disc}})$ (not continuous map)

Definition 2.11. A morphism of adic spaces $Y \longrightarrow X$ is *locally (weakly) of finite type* if locally on X and Y it is given by a morphism of affinoids induced by a homomorphism of Huber pairs that is (weakly) of topologically finite type. It is of (weakly) finite type if in addition it is quasi-compact.

Definition 2.12. Let (k, k°) be an analytic (non-discrete) field. A *rigid analytic variety* over (k, k°) is an adic space X of finite type over (k, k°).

Definition 2.13. A Huber ring A is stably sheafy if for every A-algebra B of topologically finite type and every ring of integral elements $B^+ \subset B$, the Huber pair (B, B^+) is sheafy. An adic space is *stable* if it is locally the spectrum of stably sheafy Huber pairs.

2.2 The Fiber product

Proposition 2.14 ([Eug24],9.2)**.** Let X be a stable adic space and $f : Y \longrightarrow X$ and $g : Z \longrightarrow X$ morphisms of stable adic spaces satisfying one of the following conditions:

1. X, Y and Z are perfectoid,
2. f is locally of weakly finite type and g is adic,
3. f is locally of finite type,

Then the fiber product $Y \times_X Z$ exists in the category of adic spaces and is a stable adic space. In case (i), $Y \times_X Z$ is perfectoid.

Proof Sketch. Reduce to affinoids and maps of Huber pairs,

$$X = \mathrm{Spa}(A, A^+)$$
$$Y = \mathrm{Spa}(B, B^+)$$
$$Z = \mathrm{Spa}(C, C^+)$$

and

$$\begin{tikzcd} & (B, B^+) \\ (C, C^+) & (A, A^+) \arrow[l, "\varphi_C"'] \arrow[u, "\varphi_B"] \end{tikzcd}$$

We only consider the first two cases, in which both morphisms are adic. So there are rings of definition $A_0 \subset A$, $B_0 \subset B$, and $C_0 \subset C$, with $\varphi_B(A_0) \subset B_0$ and $\varphi_C(A_0) \subset A_0$, such that for $I_A \subset A_0$, $\varphi_B(I_A)B_0$ and $\varphi_C(I_A)C_0$. We set

$$D = B \otimes_A C$$

We define its ring of definition D_0 as the image of $B_0 \otimes_{A_0} C_0$ in D and let D^+ be the integral closure of $B^+ \otimes_{A^+} C^+$ in D. Then (D, D^+) is a Huber pair whose ideal of definition is generated by the image of I_A in D_0. The additional assumptions are needed to ensure that (D, D^+) is sheafy. It then turns out that

$$\mathrm{Spa}(D, D^+) = Y \times_X Z.$$

Example 2.15 (Two closed unit disks)**.** Take two closed unit disks $\mathbf{D}_1 = \mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$ and $\mathbf{D}_2 = \mathrm{Spa}(k\langle S\rangle, k^\circ\langle S\rangle)$, then both are of finite type over $\mathrm{Spa}(k, k^\circ)$ so the fiber product is given by the tensor product

$$\mathbf{D}_1 \times_{\mathrm{Spa}(k,k^\circ)} \mathbf{D}_2 = \mathrm{Spa}(k\langle S, T\rangle, k^\circ\langle S, T\rangle)$$

Example 2.16 (Generic fibers)**.** see [Eug24] 6.12

□

2.3 Etale and smooth

Definition 2.17. Let $f : X \longrightarrow Y$ be a morphism of locally noetherian adic spaces. Assume that f is of locally finite type. Assume that (A, A^+) is a noetherian Huber pair and that $I \subset A$ is an ideal with $I^2 = 0$. Write $(A/I)^+$ for the integral closure of A^+ inside A/I and write $S = \mathrm{Spa}(A, A^+)$ and $T = \mathrm{Spa}(A/I, (A/I)^+)$.

We say that f is smooth (etale) if, for any (A, A^+) and I as above and any morphism $S \longrightarrow Y$, any Y-morphism $T \longrightarrow X$ lifts (uniquely) to a Y-morphism $S \longrightarrow X$.

Example 2.18.

1. Etale morphisms:
   (a) inclusion of affinoids $U \hookrightarrow X$
   (b) $\mathrm{Spa}(k\langle T, T^{-1}\rangle, k^\circ\langle T, T^{-1}\rangle) \longrightarrow \mathrm{Spa}(k\langle T, T^{-1}\rangle, k^\circ\langle T, T^{-1}\rangle)$ induced by $T \longmapsto T^n$
2. Smooth but not etale morphisms:
   (a) $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle) \longrightarrow \mathrm{Spa}(k, k^\circ)$
   (b) $X \times_{\mathrm{Spa}(k,k^\circ)} Y \longrightarrow X$
3. Not smooth morphisms:
   (a) $\mathrm{Spa}(k\langle x, y\rangle/(y^2 - x^3), k^\circ\langle x, y\rangle/(y^2 - x^3)) \longrightarrow \mathrm{Spa}(k, k^\circ)$.

3 Analytification

One of the original goals of rigid analytic geometry is to provide a target category for an analogue of the analytification functor:

$$\begin{aligned}\{\text{finite type schemes over } \mathbb{C}\} &\longrightarrow \{\text{complex analytic spaces}\}\\ X &\longmapsto X^{\mathrm{an}} := X(\mathbb{C})\end{aligned}$$

for schemes over non-archimedean local fields.

For an adic space X, let $\underline{X} = (X, \mathcal{O}_X)$ denote the underlying ringed space.

Proposition 3.1 ([Eug24],10.1)**.** Let $Y \longrightarrow X$ be a morphism of schemes that is locally of finite type, Z a stable adic space, and $\underline{Z} \longrightarrow X$ a morphism of locally ringed spaces. Then the fiber product $Y \times_X Z$ exists, is a stable adic space, and the projection

$$Y \times_X Z \longrightarrow Z$$

is locally of finite type.

Remark 3.2. By fiber product we mean an adic space together with a morphisms $Y \times_X Z \longrightarrow Z$, $\underline{Y \times_X Z} \longrightarrow Y$ whose underlying locally ringed space $\underline{Y \times_X Z}$ fits into a diagram and satisfies the following universal property for all adic spaces S

$$\begin{tikzcd}
S \arrow[dr, dashed, "\exists ! \underline{g}"] \arrow[drr, bend left] \arrow[ddr, bend right, "\underline{f}"'] & & \\
& \underline{Y \times_X Z} \arrow[r] \arrow[d] & Y \arrow[d] \\
& \underline{Z} \arrow[r] & X
\end{tikzcd}$$

Proof Sketch. We may assume X and Y are affine, and Z is affinoid:

$$X = \mathrm{Spec} A \quad Y = \mathrm{Spec} A[T_1, \dots, T_n]/I$$
$$Z = \mathrm{Spa}(B, B^+).$$

We restrict to the case where B is a Tate ring with pseudouniformizer ϖ. For $k \in \mathbb{N}$ consider Tate rings

$$B\langle \varpi^k T_1, \dots, \varpi^k T_n \rangle, \quad B^+\langle \varpi^k T_1, \dots, \varpi^k T_n \rangle$$

with compatible homomorphisms

$$B\langle \varpi^m T_1, \dots, \varpi^m T_n \rangle \longrightarrow B\langle \varpi^k T_1, \dots, \varpi^k T_n \rangle$$

for $m \leq k$ and

$$B[T_1, \dots, T_n] \longrightarrow B\langle \varpi^k T_1, \dots, \varpi^k T_n \rangle$$

so we a can view $I \subset B\langle \varpi^k T_1, \dots, \varpi^k T_n \rangle$. Define

$$B_k := B\langle \varpi^k T_1, \dots, \varpi^k T_n \rangle / IB\langle \varpi^k T_1, \dots, \varpi^k T_n \rangle$$

and B_k^+ as the integral closure of $B^+\langle \varpi^k T_1, \ldots, \varpi^k T_n\rangle$ in B_k. The embeddings induce open immersions $\mathrm{Spa}(B_m, B_m^+) \hookrightarrow \mathrm{Spa}(B_k, B_k^+)$ which we can glue along the embeddings to

$$Y \times_X Z = \bigcup_k \mathrm{Spa}(B_k, B_k^+).$$

To show this indeed fulfills the universal property, we need to show for any sheafy Huber pair (C, C^+) with the outer diagram

$$\begin{tikzcd}
C & & \\
& B_k \arrow[ul, dashed, "\psi"'] & A[T_1, \ldots, T_n]/I \arrow[l] \arrow[ull, bend right] \\
& B \arrow[u] \arrow[uul, bend left, "\varphi"] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

that the dashed arrow exists. Let c_i be the images of T_i in C. For big enough k, the elements $\varpi^k c_i$ are all contained in C^+ as C^+ is open and ϖ is topologically nilpotent. Thus, we obtain a well-defined homomorphism

$$\mathrm{Spa}(B_k, B_k^+) \longrightarrow \mathrm{Spa}(C, C^+)$$

mapping $\varpi^k T_i$ to $\varpi^k c_i$, and it is clear from the construction that the corresponding diagram commutes. $\square$

Definition 3.3. Let (k, k^+) be an affinoid field. The analytification of a variety X/k is defined as

$$X^{\mathrm{an}} = X \times_{\mathrm{Spec} k} \mathrm{Spa}(k, k^+).$$

Example 3.4 (The affine line)**.** We start with $\mathbf{A}^1 = \mathrm{Spec}(k[T])$. If we followe the recipe above, we are gluing $\mathrm{Spa}(k\langle \varpi^k T\rangle, k^\circ\langle \varpi^k T\rangle)$ together, i.e. closed unit disks. We therefore obtain the adic affine line.
Similarly, the analytification of $\mathbb{P}_k^1$ is the adic projective line.

3.1 Formal Schemes and analytification

Recall that there is an equivalence of categories ([Eug24], 8.8)

$$\begin{array}{c}\{\text{qc } \varpi\text{-torsion free formal } k\text{-schemes} \\ \text{of tft localized by admissible formal blowups}\}\end{array} \xrightarrow{\sim} \{\text{qcqs rigid } k\text{-spaces}\}$$

$$\mathfrak{X} \longmapsto \mathfrak{X}_\eta^{\mathrm{ad}}$$

sending a formal scheme (identified with the associated adic space $\mathfrak{X}^{\mathrm{ad}}$) to its generic fiber. So starting from a scheme X of finite type over a non-archimedean local field k we can:

1. Take some model $\mathcal{X}$ over k°, and then complete, to obtain a formal scheme $\mathfrak{X}$.
2. Identify with the adic space $\mathfrak{X}^{\mathrm{ad}}$ and take the generic fiber $\mathfrak{X}^{\mathrm{ad}}_\eta$

so this construction also gives us a rigid analytic k-space.

So given a scheme X of finite type over k we have to associated rigid analytic spaces X^{an} and $\mathfrak{X}^{\mathrm{ad}}_\eta$. When do they agree?

Example 3.5 (The affine line)**.** Choose $X = \mathbf{A}^1_k$.

1. We saw that X^{an} is the adic affine line $\mathbf{A}^1_{(k,k^\circ)}$, obtained by gluing closed disks $\mathrm{Spa}(k\langle\varpi^k T\rangle, k^\circ\langle\varpi^k T\rangle)$.
2. A formal model is given by $\mathrm{Spec}(k^\circ[T])$, with completion the formal scheme $\mathrm{Spf}(k^\circ\langle T\rangle)$ which has generic fiber $\mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$, the closed unit disk.

So these two disagree!

Theorem 3.6 ([Con07], 3.3.9)**.** For X a separated finite type k-scheme, there is a functorial quasi-compact open immersion of rigid spaces $i_X : \mathfrak{X}_\eta \longrightarrow X^{\mathrm{an}}$ that is compatible with fiber products. It is an isomorphism when X is k-proper.

Theorem 3.7 (GAGA, [Eug24], 3.19)**.** Let S be a proper algebraic variety over K, with analytification S^{an}. Given a coherent sheaf $\mathcal{F}$ on S, there is a functorial analytification $\mathcal{F}^{\mathrm{an}}$, which is coherent sheaf. Then one has the following:

1. The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is an equivalence of the abelian categories of coherent sheaves on S and on S^{an}, respectively.
2. The two δ-functors $\mathcal{F} \mapsto H^i(S, \mathcal{F})$ and $\mathcal{F} \mapsto H^i(S^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ from coherent sheaves on S to K-vector spaces are isomorphic.

References

[Con07] Brian Conrad. "Several approaches to non-archimedean geometry". In: (2007).

[Eug24] Otmar Venjakob Eugen Hellmann Judith Ludwig. *Non-Archimedean Geometry and Eigenvarieties.* AMS, 2024.