THE EIGENVARIETY MACHINE I: TOOLKIT FROM FUNCTIONAL ANALYSIS

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Abstract. The content of this note is based on [HLV24, Section 5.2 and 5.4].

Contents

1. Background from functional analysis

Definition 1.1. Let A be a ring. A *seminorm* on A is a map $|\cdot| : A \to \mathbb{R}_{\geq 0}$ satisfying the following conditions for all $a, b \in A$:

(1) $|0| = 0$, $|1| = 1$;

(2) $|a + b| \leq \max(|a|, |b|)$ (*the ultrametric triangle inequality*);

(3) $|ab| \leq |a||b|$ (*submultiplicativity*).

A *norm* on A is a seminorm such that $|a| = 0$ implies $a = 0$. A ring A equipped with a (semi)norm is called a (semi)normed ring. A *Banach ring* is a normed ring that is complete with respect to its norm.

Remark 1.2. By definition, the norm is part of the data of a Banach ring. However, we remark that the construction of eigenvarieties depends only on the topology of the rings involved, not on the specific norm that induces it.

Definition 1.3. Let A be a normed ring. A nonzero element $a \in A$ is called *multiplicative* if $|ab| = |a||b|$ for all $b \in A$.

Definition 1.4. A *Banach-Tate ring* is a complete normed ring A that contains a unit ϖ such that $|\varpi| < 1$ and ϖ is multiplicative.

Example 1.5. Let K be a non-archimedean field and $n \in \mathbb{N}$. The classical Tate algebra $\mathcal{T}_n := K\langle X_1, \ldots, X_n\rangle$, in n variables and equipped with the Gauss norm (which is multiplicative), is a Banach-Tate ring.

Remark 1.6. We relate Banach-Tate rings to the rings underlying (analytic) adic spaces, thereby justifying the terminology. Let A be a complete Tate ring with ring of definition A_0 and pseudo-uniformizer $\varpi \in A_0$, and let p be a prime number. Then we can define a norm by setting, for any $a \in A$,

$$|a| := \inf\{p^{-n} : a \in \varpi^n A_0,\ n \in \mathbb{Z}\}, \tag{1.1}$$

in which ϖ is multiplicative. In the following, we use the term *normed Tate ring* to mean a Tate ring whose topology is also given by a norm. For a complete Tate ring, we refer to the norm of equation (1.1), attached to a chosen pseudo-uniformizer, as a *standard norm*.

Moreover, given a Banach-Tate ring A, the underlying topological ring is a complete normed Tate ring: the unit ball $\{a \in A : |a| \leq 1\}$ is a ring of definition, and ϖ is a pseudo-uniformizer. Conversely, as noted above, any complete Tate ring gives rise to a normed ring with unit ϖ such that $|\varpi| < 1$ and such that ϖ is multiplicative. Thus, we may think of a Banach-Tate ring as a complete normed Tate ring with a chosen multiplicative pseudo-uniformizer.

Example 1.7. Consider the Huber ring $\mathbb{Z}_p[\![T]\!]$ with the (p,T)-adic topology. On $X := \mathrm{Spa}(\mathbb{Z}_p[\![T]\!], \mathbb{Z}_p[\![T]\!])$, consider the rational subset $U = X\left(\frac{p,T}{T}\right) = \{x \in X : |p(x)| \leq |T(x)| \neq 0\}$. Then

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = \left(\mathbb{Z}_p\widehat{[\![T]\!][1/T]},\ \mathbb{Z}_p\widehat{[\![T]\!][p/T]}\right),$$

where both completions are taken with respect to the T-adic topology. In this case, $\mathcal{O}_X(U)$ is a Tate ring with pseudo-uniformizer T, and it becomes a Banach-Tate ring when equipped with the norm $|f| = \sup_i |a_i|r^i$ for $p^{-1} < r < 1$ and $f = \sum a_i T^i \in \mathcal{O}_X(U)$.

Definition 1.8. Let A be a normed ring. A *normed A-module* is an A-module M equipped with a function $\|\cdot\| : M \to \mathbb{R}_{\geq 0}$ such that for all $m, n \in M$ and all $a \in A$:

(1) $\|m\| = 0$ if and only if $m = 0$;

(2) $\|m + n\| \leq \max(\|m\|, \|n\|)$;

(3) $\|am\| \leq |a| \cdot \|m\|$.

If A is a Banach ring and M is complete, we say that M is a *Banach A-module*. A *Banach A-algebra B* is a Banach ring equipped with the structure of a Banach A-module.

Definition 1.9. Let A be a normed Tate ring. An A-linear map $\phi : M \to N$ between normed A-modules is called *bounded* if there exists $C \in \mathbb{R}_{\geq 0}$ such that $\|\phi(m)\| \leq C\|m\|$ for all $m \in M$.

Lemma 1.10 ([BGR84, Section 2.8.1])**.** *Let A be a normed Tate ring with a multiplicative pseudo-uniformizer, and let M and N be normed A-modules. Then an A-linear map $\phi : M \to N$ is continuous if and only if it is bounded.*

Definition 1.11. (1) Let A be a normed Tate ring, and let M and N be normed A-modules. A *homomorphism* $\phi : M \to N$ is defined to be a continuous A-linear map.

(2) The *norm* of a homomorphism $\phi : M \to N$ is defined by

$$\|\phi\| := \sup_{m \neq 0} \frac{\|\phi(m)\|}{\|m\|}.$$

Remark 1.12. If A is a Banach-Tate ring and M and N are Banach A-modules, then this norm turns the A-module $\mathrm{Hom}_A(M, N)$ of continuous A-linear homomorphisms into a Banach A-module.

Theorem 1.13 (Open Mapping Theorem, [Mor19, Theorem II.4.1.1])**.** *Let A be a Banach-Tate ring, and let M and N be Banach A-modules. Then any surjective continuous A-linear map $\phi : M \to N$ is open.*

Lemma 1.14. *Let M be a Banach A-module and P a finite Banach A-module. Then any abstract A-module homomorphism $\phi : P \to M$ is continuous.*

Proof. Let $\pi : A^r \to P$ be a surjection of A-modules, and equip A^r with its usual Banach A-module norm. Note that π is bounded and hence continuous by Lemma 1.10. By Theorem 1.13, π is open. Furthermore, $\phi \circ \pi$ is bounded and therefore also continuous. Hence, ϕ is continuous. $\square$

Remark 1.15. Recall that two norms $|\cdot|_1$ and $|\cdot|_2$ on a ring A are called equivalent if they induce the same topology, and called bounded-equivalent if for any $a \in A$, there exist constants $C_1, C_2 > 0$ such that $C_1|a|_1 \leq |a|_2 \leq C_2|a|_1$. Note that for Banach algebras over a non-archimedean field, bounded equivalence is the same as equivalence. But for Banach-Tate rings, bounded equivalence implies equivalence, but not vice versa. Here is an example: take $A = \mathbb{Z}_p$ and define $|a|_1 = p^{-v_p(a)}$, $|a|_2 = p^{-2v_p(a)}$. See [JN19, Lemma 2.1.7] for cases where the converse holds.

Let $(A, |\cdot|)$ be a Banach–Tate ring and let M be a Banach A-module, equipped with two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$, i.e., two norms that induce the same topology on M, or equivalently, such that the identity maps $\mathrm{id} : (M, \|\cdot\|_1) \to (M, \|\cdot\|_2)$ and $\mathrm{id} : (M, \|\cdot\|_2) \to (M, \|\cdot\|_1)$ are continuous. By Lemma 1.10, the identity maps are bounded, and thus there exist two positive constants C_1 and C_2 such that $C_1\|m\|_2 \leq \|m\|_1 \leq C_2\|m\|_2$ for all $m \in M$. Let $\phi : M \to M$ be a continuous A-linear map. Then for all $m \in M$, we have $C_1\|\phi(m)\|_2 \leq \|\phi(m)\|_1 \leq \|\phi\|_1 \cdot \|m\|_1 \leq C_2\|\phi\|_1 \cdot \|m\|_2$, which implies that $\|\phi\|_2 \leq (C_2/C_1)\|\phi\|_1$. Similarly, we have $\|\phi\|_1 \leq (C_2/C_1)\|\phi\|_2$, which means the operator norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathrm{Hom}_A(M, M)$ are equivalent.

Notation 1.16. Let I be a set, and A a Banach-Tate ring. Given a map $I \to A$, written $i \mapsto a_i$, we write $\lim_{i\to\infty} a_i = 0$ to mean that for any $\varepsilon > 0$, there are only finitely many $i \in I$ such that $|a_i| > \varepsilon$.

Definition 1.17. Let A be a Banach-Tate ring. A Banach A-module M is called *orthonormalisable* (*ONable* for short) if there exists a subset $\{e_i : i \in I\}$ of M such that every element $m \in M$ can be uniquely written as $m = \sum_{i\in I} a_i e_i$ with $a_i \in A$, $\lim_{i\to\infty} a_i = 0$, and $\|m\| = \max_{i\in I} |a_i|$. Such a subset $\{e_i : i \in I\}$ is called an *orthonormal basis* (*ON basis* for short) of M.

Example 1.18. The classical Tate algebra $K\langle T\rangle$ over a non-archimedean field K is orthonormalisable with an orthonormal basis given by $\{T^i : i \in \mathbb{N}\}$.

Remark 1.19. Note that the second condition of Definition 1.17 implies that $\|e_i\| = 1$ for all $i \in I$. Let $c_A(I)$ be the A-module of functions $f : I \to A$ such that $\lim_{i\in I} f(i) = 0$, with addition and A-action defined pointwise. The norm $\|f\| = \max_{i\in I} |f(i)|$ gives it a Banach A-module structure which is also ONable, with canonical ON basis $e_i : I \to A$ defined by $e_i(j) = 1$ if $i = j$ and $e_i(j) = 0$ otherwise.

Let $\phi : M \to N$ be a homomorphism of ONable Banach A-modules M and N, with ON bases $\{e_i : i \in I\}$ and $\{f_j : j \in J\}$, respectively. We define the matrix $(a_{i,j})_{i\in I, j\in J}$ associated to ϕ by writing $\phi(e_i) = \sum_{j\in J} a_{i,j} f_j$. It is easy to check that $\|\phi\| = \sup_{i,j} |a_{i,j}|$, and that this matrix satisfies: (1) $\lim_{j\to\infty} a_{i,j} = 0$ for all $i \in I$, and (2) $|a_{i,j}| \le C$ for some $C \in \mathbb{R}$ and all i, j. Conversely, given a collection $(a_{i,j})$ of elements in A satisfying (1) and (2), there exists a unique continuous map $\phi : M \to N$ with norm $\|\phi\| = \sup_{i,j} |a_{i,j}|$ associated to $(a_{i,j})$. In particular, we can measure the distance between two homomorphisms via their matrices. For $\phi, \psi \in \mathrm{Hom}_A(M, N)$ with matrices $(a_{i,j})$ and $(b_{i,j})$, respectively, we have $\|\phi - \psi\| \le \varepsilon$ if and only if $|a_{i,j} - b_{i,j}| \le \varepsilon$ for all i, j.

Definition 1.20. Let A be a Banach-Tate ring and let M and N be Banach A-modules. A homomorphism $\phi : M \to N$ is said to be of *finite rank* if its image is contained in a finitely generated A-submodule of N, and is called *compact* if it is the limit of finite-rank homomorphisms in the Banach A-algebra $\mathrm{Hom}_A(M, N)$.

Lemma 1.21 ([Mor19, Proposition II.4.2.2])**.** *Let A be a Noetherian Banach-Tate ring. Every finitely generated A-module M admits (up to equivalence) a unique norm making it into a Banach A-module. Any A-linear map $f : M \to N$ between finite Banach A-modules is continuous; its image $f(M) \subseteq N$ is closed, and the induced map $M \to f(M)$ is open.*

Remark 1.22. From now on, assume A is a Noetherian Banach-Tate ring. Let M be an ONable Banach A-module with ON basis $\{e_i : i \in I\}$. If $S \subseteq I$ is finite, we define A^S to be the submodule $\bigoplus_{i\in S} Ae_i \subseteq M$. Consider the projection

$$\pi_S : M \to A^S, \quad \sum_{i\in I} a_i e_i \mapsto \sum_{i\in S} a_i e_i.$$

It is natural to interpret the notions of finite-rank and compact morphisms in terms of compositions with π_S.

Lemma 1.23 ([Buz07, Lemma 2.3])**.** *Let M be an ONable Banach A-module with ON basis $\{e_i : i \in I\}$, and let P be a finite A-submodule of M.*

(1) *There exists a finite subset $S \subseteq I$ such that the projection $\pi_S : M \to A^S$ is injective when restricted to P.*

(2) *P is a closed subset of M, and hence is complete.*

(3) *For all $\varepsilon > 0$, there exists a finite subset $T \subseteq I$ such that for all $p \in P$, we have $\|\pi_T(p) - p\| \le \varepsilon\|p\|$.*

Proposition 1.24. *Let M and N be ONable Banach A-modules with ON bases $\{e_i : i \in I\}$ and $\{f_j : j \in J\}$, respectively. Let $\phi : M \to N$ be a continuous A-module homomorphism with matrix $(a_{i,j})$. Then ϕ is compact if and only if* $\lim_{j\to\infty} \sup_{i\in I} |a_{i,j}| = 0$.

Proof. Assume that $\lim_{j\to\infty} \sup_{i\in I} |a_{i,j}| = 0$. Then for any $\varepsilon > 0$, there exists a finite subset $S \subseteq J$ such that $\sup_{j\notin S} \sup_{i\in I} |a_{i,j}| \le \varepsilon$, which implies that $\|\phi - \pi_S \circ \phi\| \le \varepsilon$. Hence, ϕ is a limit of finite-rank operators, i.e., compact. Conversely, suppose ϕ is compact. It suffices to prove the claim for ϕ of finite rank. If $\phi = 0$, the claim is clear. Otherwise, assume $\phi \ne 0$ and $\phi(M) \subseteq P$ for some finitely generated A-submodule $P \subseteq N$. By Lemma 1.23(3), for any $\varepsilon > 0$, we can choose a finite subset $T \subseteq J$ such that for all $p \in P$, we have $\|\pi_T(p) - p\| \le \varepsilon\|p\|/\|\phi\|$, and hence $\|\pi_T \circ \phi - \phi\| \le \varepsilon$. This implies that $|a_{i,j}| \le \varepsilon$ for all $j \notin T$ and all $i \in I$, and since ε is arbitrary, we conclude that $\lim_{j\to\infty} \sup_{i\in I} |a_{i,j}| = 0$. ☐

Example 1.25. Consider the Banach $\mathbb{Q}_p$-algebra $\mathbb{Q}_p\langle pT\rangle = \{\sum_n a_n(pT)^n : \lim_{n\to\infty} a_n = 0\} \subseteq \mathbb{Q}_p[\![T]\!]$ equip with the norm $\|\sum_n a_n(pT)^n\| = \max_n |a_n|$. By Proposition 1.24, the restriction map

$$\mathrm{res} : \mathbb{Q}_p\langle\, pT\rangle \to \mathbb{Q}_p\langle T\rangle, \quad \sum_n a_n(pT)^n \mapsto \sum_n (a_n p^n)T^n$$

is a compact operator. (Here, $\mathbb{Q}_p\langle T\rangle$ equipped with the Gauß norm.)

Construction 1.26. Let M be a Banach A-module with ON basis $\{e_i : i \in I\}$, and let $\phi : M \to M$ be a compact endomorphism with matrix $(a_{i,j})$. For a finite subset $S \subseteq I$, define $c_S := \sum_{\sigma: S\to S} \operatorname{sgn}(\sigma) \prod_{i\in S} a_{i,\sigma(i)}$, where the sum runs over all permutations σ of S. For $n \geq 0$, define

$$c_n := (-1)^n \sum_{S\subseteq I,\ |S|=n} c_S.$$

By Proposition 1.24, each c_n is well-defined in A[1]. We define the *characteristic power series* (also called the *Fredholm determinant*) of ϕ as

$$\det(1 - X\phi) := \sum_{n\geq 0} c_n X^n \in A[\![X]\!].$$

Remark 1.27. When M is a finite-dimensional vector space over a field K, the Fredholm determinant $\det(1-X\phi)$ agrees with the usual algebraic characteristic polynomial of ϕ.

Proposition 1.28. *Let $\phi : M \to M$ be a compact endomorphism on an ONable Banach A-module M with ON basis $\{e_i : i \in I\}$. Then the characteristic power series $\det(1 - X\phi) = \sum_n c_n X^n \in A[\![X]\!]$ converges for all $X \in A$.*

Proof. Let $r_1, r_2, \ldots$ be the sequence of real numbers obtained by arranging the set $\{\sup_{i\in I} |a_{i,j}| : j \in I\}$ in decreasing order. By Proposition 1.24, we have $\lim_{k\to\infty} r_k = 0$. If $S \subseteq I$ with $|S| = n$, then each product $a_{S,\sigma} := \prod_{i\in S} a_{i,\sigma(i)}$ satisfies $|a_{S,\sigma}| \leq r_1 \cdots r_n$, and hence $|c_n| \leq r_1 \cdots r_n$. If $\alpha \in \mathbb{R}_{>0}$ is any positive real number, then $|c_n|\alpha^n \leq (r_1\alpha)\cdots(r_n\alpha)$. Since $r_i\alpha \to 0$, the product tends to zero as $n \to \infty$. Hence $|c_n|\alpha^n \to 0$, which shows the claim. □

Lemma 1.29. *Let M be an ONable Banach A-module with ON basis $\{e_i : i \in I\}$.*

(1) *If $\phi_n : M \to M$, $n \in \mathbb{N}_{>0}$, is a sequence of compact operators that converges to a compact operator ϕ, then $\lim_n \det(1 - X\phi_n) = \det(1 - X\phi)$, uniformly in the coefficients (with respect to the norm of A).*

(2) *If $\phi : M \to M$ is compact and the image of ϕ is contained in A^S for a finite subset $S \subseteq I$, then $\det(1-X\phi) = \det(1 - X\phi|_{A^S})$, where the right-hand side is the usual algebraically defined determinant.*

(3) *If $\phi : M \to M$ is compact and the image of ϕ is contained in an arbitrary finite A-submodule $Q \subseteq M$, which is free of finite rank, then $\det(1-X\phi) = \det(1-X\phi|_Q)$, where again the right-hand side is the usual algebraically defined determinant.*

Proof. We only prove (1); for the proofs of (2) and (3), see [Buz07, Lemma 2.5]. Let $(a_{i,j})$ and $(a^{(n)}_{i,j})$ be the matrices of ϕ and ϕ_n, respectively, and write $\det(1 - X\phi) = \sum_m c_m X^m$, $\det(1 - X\phi_n) = \sum_m c^{(n)}_m X^m$. Let $0 < \varepsilon < 1$ and let $r_1, \ldots, r_h$ be the elements of $\{\sup_{i\in I} |a_{i,j}| : j \in I\}$ greater than 1. Choose $\eta \in \mathbb{R}_{>0}$ such that $\eta\, r_1 \cdots r_h \leq \varepsilon$. Pick n large enough so that $\|\phi - \phi_n\| \leq \eta$. Then $|a_{i,j} - a^{(n)}_{i,j}| \leq \eta$ for all i, j. In particular, $\sup_{i\in I} |a_{i,j}| = \sup_{i\in I} |a^{(n)}_{i,j}|$ if $\sup_{i\in I} |a_{i,j}| > \eta$, and $\sup_{i\in I} |a_{i,j}| \leq \eta$ otherwise. For a finite set $S \subseteq I$ equipped with an order, we have

$$\begin{aligned} |a_{S,\sigma} - a^{(n)}_{S,\sigma}| &= \left| \prod_{i\in S} a_{i,\sigma(i)} - \prod_{i\in S} a^{(n)}_{i,\sigma(i)} \right| = \left| \sum_{i\in S} \left(a_{i,\sigma(i)} - a^{(n)}_{i,\sigma(i)}\right) \prod_{j<i} a_{j,\sigma(j)} \prod_{k>i} a^{(n)}_{k,\sigma(k)} \right| \\ &\leq \eta \sup_{i\in S} \prod_{\substack{j\in S\\ j\neq i}} \sup\left(\sup_{i\in I} |a_{i,j}|,\ \sup_{i\in I} |a^{(n)}_{i,j}| \right). \end{aligned}$$

Hence $|a_{S,\sigma} - a^{(n)}_{S,\sigma}| \leq \eta\, r_1 \cdots r_h \leq \varepsilon$. Summing over all subsets S of size m, we obtain $|c_m - c^{(n)}_m| \leq \varepsilon$ for all m, proving uniform convergence of the coefficients. □

Proposition 1.30. *Let $(M, \|\cdot\|_1)$ and $(M, \|\cdot\|_2)$ be ONable Banach A-modules. Suppose that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on M, i.e., they induce the same topology and both turn M into an ONable Banach A-module. Let $\phi : M \to M$ be an A-linear endomorphism. Then ϕ is compact with respect to $\|\cdot\|_1$ if and only if it is compact with respect to $\|\cdot\|_2$. Furthermore, if $\{e_i : i \in I\}$ and $\{f_j : j \in J\}$ are ON bases of $(M, \|\cdot\|_1)$ and $(M, \|\cdot\|_2)$, respectively, then the characteristic power series $\det(1 - X\phi)$ with respect to both bases coincide.*

[1] Check that for any $\varepsilon > 0$, there are only finitely many subsets S with $|c_S| > \varepsilon$, which implies that the family $\{c_S\}$ is summable.

Proof. By Remark 1.15, the operator norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathrm{Hom}_A(M,M)$ are equivalent. The first claim follows since the set of compact operators depends only on the topology of $\mathrm{Hom}_A(M,M)$. As in the proof of Proposition 1.24, we may find finite subsets (T_n) such that $\|\pi_{T_n}\circ\phi-\phi\|_1\to 0$ and $\|\pi_{T_n}\circ\phi-\phi\|_2\to 0$ as $n\to\infty$. Applying Lemma 1.29(1) and (3) to the limit $\phi=\lim_{n\to\infty}\pi_{T_n}\circ\phi$, the result follows. $\square$

Corollary 1.31. *Let $(M,\|\cdot\|_1)$ be an ONable Banach $(A,|\cdot|_1)$-module and let $(M,\|\cdot\|_2)$ be an ONable Banach $(A,|\cdot|_2)$-module. Suppose the norms $|\cdot|_1$ and $|\cdot|_2$ are equivalent on A, and the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on M. Let $\phi: M\to M$ be an A-linear endomorphism. Then ϕ is compact with respect to $\|\cdot\|_1$ if and only if it is compact with respect to $\|\cdot\|_2$. Furthermore, the characteristic power series* $\det(1-X\phi)$ *with respect to both norms coincide.*

Proof. This follows immediately from Proposition 1.30. $\square$

Proposition 1.32 ([Buz07, Lemma 2.7])**.** *If M and N are ONable Banach A-modules, and if $\phi: M\to N$ is compact and $\psi: N\to M$ is continuous, then the compositions $\psi\circ\phi$ and $\phi\circ\psi$ are compact, and* $\det\big(1-X(\phi\circ\psi)\big)=\det\big(1-X(\psi\circ\phi)\big)$.

Definition 1.33. (1) A Banach A-module M is called *potentially ONable* if there exists a set I such that M is A-linearly isomorphic to $c_A(I)$. A set in M corresponding to $\{e_i=(\delta_{i,j})_j : i\in I\}$ under such an isomorphism is called a *potential ON basis.*

(2) We say that a Banach A-module has *property* (Pr) if it is a direct summand of a potentially ONable Banach A-module.

Remark 1.34. The notions of being potentially ONable and of having property (Pr) are independent of the chosen norms: they depend only on the underlying topological A-module structure. In comparison, a Banach A-module M is ONable if and only if M is A-linearly isometric to $c_A(I)$ for some I.

Example 1.35. (1) Let $A=\mathbb{Q}_p$ and $M=\mathbb{Q}_p(\sqrt{p})$, equipped with its usual norm. Since $\|M\|\neq|A|$, M is not ONable but is potentially ONable.

(2) Let K be a discretely valued non-archimedean field. Then any Banach space over K is potentially ONable (cf. [Ser62, Proposition 1]).

Proposition 1.36. *A Banach A-module P has property* (Pr) *if and only if for every surjection $f: M\to N$ of Banach A-modules and for every continuous map $\alpha: P\to N$, the map α lifts to a map $\beta: P\to M$ such that $f\circ\beta=\alpha$.*

Proof. If $P=c_A(I)$ for some set I, then to give α is to give a bounded map $\alpha': I\to N$, i.e. $\sup_{i\in I}\|\alpha'(i)\|_N<\infty$, and such a map lifts to a bounded map $I\to M$ by the open mapping theorem (Theorem 1.13), which gives β. Conversely, the claim is clear by choosing a surjective $f: c_A(I)\to P$ for some set I and taking $\alpha=\mathrm{id}_P$. $\square$

Remark 1.37. When P is finitely generated, property (Pr) is equivalent to P being a projective A-module.

Construction 1.38. Let M and B be normed A-modules. Define a function

$$|\cdot|: M\otimes_A B\longrightarrow\mathbb{R}_{\geq 0},\quad |g|=\inf\left\{\max_{1\leq i\leq r}\big(\|m_i\|\cdot|b_i|\big)\right\},$$

where the infimum runs over all representations $g=\sum_{i=1}^r m_i\otimes b_i$ with $m_i\in M$ and $b_i\in B$. This turns $M\otimes_A B$ into a seminormed A-module. Its completion is denoted by $M\widehat{\otimes}_A B$ and is a normed A-module. If $h: A\to B$ is a contractive morphism of Banach-Tate rings (i.e., $|h(a)|\leq|a|$ for all $a\in A$), then B is a normed A-module and $M\widehat{\otimes}_A B$ is a normed B-module. In particular, given two complete Tate rings A and B and a continuous map $h: A\to B$, the image of a pseudo-uniformizer is a pseudo-uniformizer. If we choose a pseudo-uniformizer $\varpi_A\in A$ and compatible rings of definition, then equipping A and B with their standard norms makes h contractive.

Lemma 1.39. *Let $h: A\to B$ be a continuous morphism of noetherian Banach-Tate algebras, and equip A and B with their standard norms. Then:*

(1) *If M is a potentially ONable Banach A-module, then $M\widehat{\otimes}_A B$ is a potentially ONable Banach B-module. Furthermore, if $\{e_i : i\in I\}$ is a potential ON basis of M, then $\{e_i\otimes 1_B : i\in I\}$ is a potential ON basis of $M\widehat{\otimes}_A B$.*

(2) *If M has property* (Pr)*, then so does $M \widehat{\otimes}_A B$.*

Proof. Note that (2) follows from (1). For (1), set $N = c_B(I)$ and let $\{f_i : i \in I\}$ be its canonical ON basis. The natural A-bilinear bounded map $M \times B \to N$ sending $(\sum_i a_i e_i, b)$ to $\sum_i bh(a_i) f_i$ induces a continuous map $\phi : M \widehat{\otimes}_A B \to N$. On the other hand, for $n \in N$ written as a limit of elements of the form $\sum_{i \in S} b_i f_i$ with $S \subseteq I$ finite, observe that $\left|\sum_{i\in S} e_i \otimes b_i\right| \leq C \max_{i \in S} |b_i|$ for some constant $C > 0$. Hence, as S grows, the resulting sequence is Cauchy, and its image in $M \widehat{\otimes}_A B$ converges to a limit, giving a continuous map $N \to M \widehat{\otimes}_A B$ which is inverse to ϕ. □

Corollary 1.40. *Let $h : A \to B$ be a continuous morphism of noetherian Banach-Tate algebras.*

(1) *Let M and N be potentially ONable Banach A-modules with potential ON bases $\{e_i : i \in I\}$ and $\{f_j : j \in J\}$, respectively. If $\phi : M \to N$ is compact with matrix $(a_{i,j})$, then $\phi \otimes 1 : M \widehat{\otimes}_A B \to N \widehat{\otimes}_A B$ is compact. Moreover, if $(b_{i,j})$ is the matrix of $\phi \otimes 1$ with respect to the bases $\{e_i \otimes 1 : i \in I\}$ and $\{f_j \otimes 1 : j \in J\}$, then $b_{i,j} = h(a_{i,j})$ for all $i \in I$, $j \in J$.*

(2) *If M and N have property* (Pr)*, then $\phi \otimes 1$ is compact.*

(3) *If M has property* (Pr) *and ϕ is a compact endomorphism with* $\det(1 - X\phi) = \sum_n c_n X^n$*, then* $\det(1 - X(\phi \otimes 1)) = \sum_n h(c_n) X^n$.

Proof. The relation $b_{i,j} = h(a_{i,j})$ follows directly from the definition. Compactness follows from Proposition 1.24. Claim (3) then follows from (1) and (2). □

2. Riesz theory

Definition 2.1. Let A be a noetheroan Banach-Tate ring. The ring $A\{\{X\}\}$ of *entire power series* is defined as

$$A\{\{X\}\} := \left\{ \sum_n a_n X^n \in A[\![X]\!] : \text{for all } r \in \mathbb{Z},\ \lim_{n\to\infty} a_n \varpi^{rn} = 0 \right\}.$$

Remark 2.2. The characteristic power series $\det(1 - X\phi)$ of a compact endomorphism $\phi : M \to M$ of a Banach A-module M with property (Pr) is an entire power series.

Construction 2.3. Let A be a noetherian Banach-Tate ring, and let $Q \in A[X]$ be a polynomial. Define $Q^*(X) := X^{\deg Q} Q(1/X) \in A[X]$. For an entire power series $S \in A\{\{X\}\}$, we define the *resultant* $R(S, Q) \in A$ as follows: write $Q(X) = X^n - a_1 X^{n-1} + \cdots + (-1)^n a_n$ and let $e_1, \ldots, e_n$ be the elementary symmetric polynomials in variables $T_1, \ldots, T_n$. Then there exists a unique $H \in A\{\{T_1, \ldots, T_n\}\}$ such that $H(e_1, \ldots, e_n) = S(T_1) \cdots S(T_n)$. We then define $R(S, Q) := H(a_1, \ldots, a_n)$. This extends the classical resultant of polynomials. By [Col97, Lemma A3.7], $R(S, Q)$ is a unit if and only if S and Q are relatively prime.

Lemma 2.4 ([Col97, Lemma A4.1])**.** *Let A be a noetherian Banach-Tate ring, and let M be a Banach A-module with property (Pr) and a compact endomorphism ϕ with characteristic power series $F(X) = \det(1 - X\phi)$. If $Q \in A[X]$ is a monic polynomial, then Q and F are relatively prime in $A\{\{X\}\}$ if and only if $Q^*(\phi)$ is an invertible operator on M.*

Definition 2.5. Let A be a noetherian Banach-Tate ring.

(1) Let M be a Banach A-module with property (Pr) and let ϕ be a compact endomorphism with characteristic power series F. The *Fredholm resolvent* of ϕ is defined as $R_F(X) := F(X)/(1 - X\phi) \in A[\phi][\![X]\!]$.

(2) Let $k \in \mathbb{N}$. For any power series $f = \sum_{n\geq 0} a_n X^n \in R[\![X]\!]$ over a ring R and $s \in \mathbb{N}$, define $\Delta^s f := \sum_{n\geq 0} \binom{n+s}{s} a_{n+s} X^n \in R[\![X]\!]$. We say that $a \in A$ is a *zero of order* k of $H \in A\{\{X\}\}$ if $(\Delta^s H)(a) = 0$ for all $s < k$ and $(\Delta^k H)(a)$ is a unit.

Remark 2.6. By [Ser62, Proposition 10], if $R_F(X) = \sum_{m\geq 0} v_m X^m$ with $v_m \in \mathrm{End}_A(M)$, then for all $r \in \mathbb{R}_{>0}$, we have $\lim_{m\to\infty} \|v_m\| r^m = 0$.

For Δ^s, it is straightforward to verify that it maps $A\{\{X\}\}$ to itself. Let $k \geq 1$ and write $H = 1 + a_1 X + \cdots$. Then $\Delta^0 H(a) = 0$ implies that $-1 = a(a_1 + a_2 a + \cdots)$ which in turn implies that $a \in A^\times$. By induction on k, we can show that $H(X) = (1 - a^{-1} X)^k G(X)$ where $G \in A\{\{X\}\}$ satisfies $G(a) \in A^\times$.

Proposition 2.7. *Let M be a Banach A-module with property* (Pr) *and let ϕ be a compact endomorphism with characteristic power series $F(X)$. Let $a \in A$ be a zero of order k of $F(X)$. Then there is a unique decomposition $M = N \oplus M'$ into closed ϕ-stable submodules such that $1 - a\phi$ is invertible on M' and $(1-a\phi)^k = 0$ on N. The submodules N and M' are defined, respectively, as the kernel and image of a projector lying in the closure (in $\mathrm{End}_A(M)$) of $A[\phi]$. Moreover, N is projective of rank k, and assuming $k \geq 1$, we have that a is a unit and the characteristic power series of ϕ restricted to N is $(1 - a^{-1}X)^k$.*

Proof. Let $f_s = \Delta^s R_F(a)$. Note that $(1 - X\phi)R_F(X) = F(X)$ in $A[\phi][\![X]\!]$. Applying Δ^s, we obtain $(1 - X\phi)\Delta^s R_F(X) - \phi\,\Delta^{s-1}R_F(X) = \Delta^s F(X)$. Putting $X = a$, we have the following relations:

$$\begin{aligned}
(1-a\phi)f_0 &= 0,\\
(1-a\phi)f_1 - \phi f_0 &= 0,\\
&\vdots\\
(1-a\phi)f_{k-1} - \phi f_{k-2} &= 0,\\
(1-a\phi)f_k - \phi f_{k-1} &= c,
\end{aligned}$$

with $c \in A^\times$. So for $s < k$, we have $(1-a\phi)^{s+1}f_s = 0$. Set $e = c^{-1}(1-a\phi)f_k$ and $f = -c^{-1}\phi f_{k-1}$. The last equation gives $e + f = 1$, and $(1-a\phi)^k f_{k-1} = 0$ implies that $fe^k = 0$. Expanding $(e+f)^k = 1$, i.e. $e^k + \big(ke^{k-1}f + \cdots + kef^{k-1} + f^k\big) = 1$, and setting $p = e^k$ and $q = ke^{k-1}f + \cdots + kef^{k-1} + f^k$, we have $p + q = 1$ and $pq = 0$. This implies that $p^2 = p$ and $q^2 = q$, which shows that p and q are projections in $\mathrm{End}_A(M)$. Note that they both lie in the closure of $A[\phi]$. Now set $N = \ker(p) = \ker(1-a\phi)^k$ and $M' = \mathrm{im}(p) = \mathrm{im}(1-a\phi)^k$[2]. Then $(1-a\phi)^k = 0$ on N and $(1-a\phi)$ is invertible on M'.

It is clear that N satisfies (Pr), but furthermore $(1-a\phi)^k = 0$ on N implies that $1 = ka\phi + \cdots + (-1)^{k-1}a^k\phi^k$, hence the identity is a compact operator on N. Thus we can choose $\alpha : N \to N$ of finite rank such that $1 - \alpha$ is sufficiently small. As $\|(1-\alpha)^n\| \to 0$, $\alpha = 1 - (1-\alpha)$ is invertible on N, which implies that N is finitely generated. By Remark 1.37, N is projective. If $k = 0$, then $N = 0$ and $M' = M$, and we are done. If $k \geq 1$, then a is a unit by Remark 2.6. Let F_N and $F_{M'}$ be the characteristic power series of ϕ restricted to N and M', respectively. Then $F = F_N F_{M'}$. By Lemma 2.4, $F_{M'}$ and $(X-a)^k$ generate the unit ideal in $A\{\{X\}\}$. Hence $(1-a^{-1}X)^k$ divides F_N in $A\{\{X\}\}$[3]. Then write $F_N = (1-a^{-1}X)^k D(X)$ for some $D \in A[X] \subseteq A\{\{X\}\}$, since N is finitely generated. From $F = (X-a)^k G(X)$, we have $D(X)$ divides $G(X)$ because $(1-a^{-1}X)^k$ has constant term 1 and hence is not a zero-divisor in $A\{\{X\}\}$. Since $G(a)$ is a unit, $D(a)$ is also a unit. Moreover, $(1-a\phi)^k = 0$ on N, i.e. $1 = \phi\big(ak + \cdots + (-1)^{k-1}a^k\phi^{k-1}\big)$, implies that $\phi|_N$ is invertible, i.e. $\det(\phi|_N) \in A^\times$. Hence, the leading coefficient of $F_N = \det(1 - \phi|_N X)$ lies in $A^\times$. Since $F_N = (1-a^{-1}X)^k D(X)$ and $a \in A^\times$, we have that the leading coefficient of $D(X)$ lies in $A^\times$. Reducing modulo any maximal ideal of A, we see that F_N must be $(1-a^{-1}X)^k$ up to a unit[4]. Therefore N has rank k and $D \in A^\times$[5]. Since $1 = F_N(0) = D(0)$, we have $D = 1$, i.e. $F_N = (1-a^{-1}X)^k$. □

Remark 2.8. In Proposition 2.7, the zeros of F correspond to the inverses of the eigenvalues of ϕ, and all eigenvectors associated with the eigenvalue a^{-1} lie in N.

Definition 2.9. (1) A polynomial $Q \in A[X]$ is called *multiplicative* if its leading coefficient is a unit, i.e. $Q^*(0) \in A^\times$.

(2) An entire power series $S = \sum_{n\geq 0} a_n X^n \in A\{\{X\}\}$ is called a *Fredholm series* if $a_0 = 1$.

Theorem 2.10. *Let A be a Noetherian Banach-Tate ring, and let M be a Banach A-module with property* (Pr). *Let ϕ be a compact endomorphism with characteristic power series $F(X)$. Assume that we have a factorization $F = QS$, where S is a Fredholm series, $Q = 1 + \cdots + a_n X^n$ is a multiplicative polynomial of degree n, and Q and S are relatively prime in $A\{\{X\}\}$. Then:*

(1) *The submodule $\ker Q^*(\phi) \subseteq M$ is finitely generated, projective, and has a unique ϕ-stable closed complement $M(Q)$ such that $Q^*(\phi)$ is invertible on $M(Q)$.*

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[2] Clearly, we have $\ker(1-a\phi)^k \subseteq \ker(p)$. For $x \in \ker(p) = \mathrm{im}(q)$, we have $(1-a\phi)^k(x) \in \mathrm{im}\big((1-a\phi)^k \circ q\big) = 0$ by $(1-a\phi)^k f_{k-1} = 0$.

[3] Here $F = F_N F_{M'} = (X-a)^k G(X)$ for some $G(X)$, and $(X-a)^k$ and $F_{M'}$ are coprime.

[4] Here we work on $N \otimes \kappa$ over a residue field κ. $(1-a\phi)^k = 0$, i.e. $b = -(1-a\phi)$ is nilpotent and upper triangular, and $\phi = a^{-1}(1+b)$. So $\det(1-tb) = 1$ for any t. Then applying det to $1 - X\phi = 1 - Xa^{-1}(1+b) = (1-a^{-1}X)\left(1 - \frac{a^{-1}X}{1-a^{-1}X}b\right) \in A[\phi][\![X]\!]$, we have $\det(1-X\phi) = (1-a^{-1}X)^{\dim_\kappa(N\otimes\kappa)}$. Thus $\det(1-X\phi) = (1-a^{-1}X)^k$ and $\dim_\kappa(N\otimes\kappa) = k$.

[5] $D(a) \in A^\times$ implies that $\bar{D}(X) \in \kappa[X]$ is coprime with $1 - a^{-1}X$, hence $\bar{D}(X) \in \kappa$. The leading coefficient of D being a unit implies that $\deg(D) = \deg(\bar{D}) = 0$.

(2) *The idempotent projectors* $M \to \ker Q^*(\phi)$ *and* $M \to M(Q)$ *lie in the closure of* $A[\phi] \subseteq \mathrm{End}_A(M)$.

(3) *The rank of* $\ker Q^*(\phi)$ *is* $\deg Q$, *and* $\det(1 - X\phi\mid_{\ker Q^*(\phi)}) = Q$.

(4) *The operator* ϕ *is invertible on* $\ker Q^*(\phi)$, *and* $\det(1 - X\phi\mid_{M(Q)}) = S$.

Proof. Consider the operator $v = 1 - Q^*(\phi)/Q^*(0)$, which is compact and whose characteristic power series has a zero at $X = 1$ of order n[6] (see [Col97, Theorem A4.3]). Applying Proposition 2.7 to v, we have $M = N \oplus M(Q)$, where N and $M(Q)$ are defined as the kernel and image of a projector lying in the closure of $A[v]$, and hence in the closure of $A[\phi]$. Both submodules N and $M(Q)$ are ϕ-stable. Furthermore, $N = \ker(1 - (1 - Q^*(\phi)/Q^*(0)))^n = \ker(Q^*(\phi)/Q^*(0))^n = \ker\big(Q^*(\phi)\big)$[7]. Thus N is projective of rank n, and $Q^*(\phi)/Q^*(0) = \big(1 - (1 - Q^*(\phi)/Q^*(0))\big)$ is invertible on $M(Q)$, hence $Q^*(\phi)$ is invertible on $M(Q)$. By Lemma 2.4, the characteristic power series of ϕ on $M(Q)$ and Q are coprime. Denote the characteristic power series of ϕ on N and $M(Q)$ by F_N and $F_{M(Q)}$, respectively. As in the proof of Proposition 2.7, we have $F_N = Q$, which establishes the first three claims. Note that $\det(\phi|_{\ker Q^*(\phi)}) = Q^*(0) \in A^\times$, so ϕ is invertible on $\ker Q^*(\phi)$. Since $Q(0) = 1$, Q is not a zero-divisor. Thus $F = QS = F_N F_{M(Q)} = QF_{M(Q)}$, which implies $Q \cdot (S - F_{M(Q)}) = 0$, hence $S = F_{M(Q)}$. □

Definition 2.11. Let A be a Banach-Tate ring with a pseudo-uniformizer ϖ_A, equipped with the standard norm.

(1) Let $F = \sum a_n X^n \in K[\![X]\!]$ be a Fredholm series over a non-archimedean field K with a fixed pseudo-uniformizer ϖ_K. The *Newton polygon* $\mathrm{NP}(F)$ of F is defined as the convex hull in the plane of the points $(n, v(a_n))$, where $v(a_n)$ denotes the valuation of the coefficient a_n. Each segment of the Newton polygon has a *slope*, and the *multiplicity* of a segment is the positive difference between the first coordinates of its endpoints. We say that a Fredholm series F has *slope* $\leq h$ (resp. $> h$) if all the slopes of $\mathrm{NP}(F)$ are $\leq h$ (resp. $> h$). If a Fredholm series F lies in $A\{\{X\}\}$, we say that F has *slope* $\leq h$ (resp. $> h$) if, for every rank-1 point $x \in \mathrm{Spa}(A, A^+)$ with residue field $\kappa(x)$, the specialization $F_x \in \kappa(x)\{\{X\}\}$ has slope $\leq h$ (resp. $> h$).

(2) Let M be an (abstract) A-module, and let ϕ be an A-linear endomorphism of M. An element $m \in M$ is said to have *slope* $\leq h$ with respect to ϕ if there exists a multiplicative polynomial $Q \in A[X]$ such that:

   (a) $Q^*(\phi)(m) = 0$;

   (b) the slope of Q is $\leq h$.

   We let $M_{\leq h} \subseteq M$ denote the subset of elements of slope $\leq h$.

Remark 2.12. Recall that for a polynomial, the slopes of $\mathrm{NP}(F)$ correspond to the valuations of its roots. For an entire power series, combining this with Weierstraß theory, the slopes control the valuations of the zeros of F.

Lemma 2.13. $M_{\leq h}$ *is an* A*-submodule of* M, *which is stable under* ϕ.

Proof. It is clear from the definition that $M_{\leq h}$ is stable under the A-action and under ϕ. To check that it is closed under addition, it suffices to show that if Q_1 and Q_2 are multiplicative polynomials of slope $\leq h$, then $Q_1 Q_2$ also has slope $\leq h$, which is a well-known fact. □

Definition 2.14. Let A be a Banach-Tate ring with a fixed multiplicative pseudo-uniformizer ϖ_A.

(1) Let $F \in A\{\{X\}\}$ be a Fredholm series and let $h \in \mathbb{R}$. A *slope* $\leq h$*-factorization* of F is a factorization $F = QS$ in $A\{\{X\}\}$, where Q is a multiplicative polynomial of slope $\leq h$ and S is a Fredholm series of slope $> h$.

(2) Let M be an A-module with an A-linear endomorphism ϕ, and let $h \in \mathbb{Q}$. A *slope* $\leq h$*-decomposition* of M is an $A[\phi]$-module decomposition $M = M_h \oplus M^h$ such that:

   (a) M_h is a finitely generated A-submodule of $M_{\leq h}$;

   (b) for every multiplicative polynomial $Q \in A[X]$ of slope $\leq h$, the map $Q^*(\phi) : M^h \to M^h$ is an isomorphism of A-modules.

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[6] For polynomials F and G, define $D(F, G) = \prod_{i=1}^{\deg G} \big(1 - XF(x_i)\big)$, where x_i are the zeros of G. Extend D to $F, G \in A\{\{X\}\}$ by passing to the limit. Then $\det(1 - vX) = D\Big(1 - \frac{Q^*(X)}{Q^*(0)}, F\Big) = D\Big(1 - \frac{Q^*(X)}{Q^*(0)}, Q\Big) \cdot D\Big(1 - \frac{Q^*(X)}{Q^*(0)}, S\Big)$. But $D(1 - Q^*(X)/Q^*(0), Q) = (1 - X)^n$, and $D(1 - Q^*(X)/Q^*(0), S)(1) = R(Q(X)/Q^*(0), S) \in A\{\{X\}\}^\times$.

[7] The constant term of $Q^*(X)$ is a unit, hence $Q^*(\phi)$ is not a zero-divisor.

Proposition 2.15. *Let M be an A-module with an A-linear endomorphism ϕ, and let $h \in \mathbb{Q}$. If M has a slope $\leq h$-decomposition $M_h \oplus M^h$, then it is unique, and $M_h = M_{\leq h}$. In particular, $M_{\leq h}$ is finitely generated over A. Write $M_{>h}$ for the unique complement. Moreover, the slope decompositions satisfy the following functorial properties:*

(1) *Let $f : M \to N$ be a morphism of $A[\phi]$-modules with slope $\leq h$-decompositions. Then $f(M_{\leq h}) \subseteq N_{\leq h}$ and $f(M_{>h}) \subseteq N_{>h}$. Further, both* $\ker(f)$ *and* $\mathrm{im}(f)$ *have slope $\leq h$-decompositions.*

(2) *Let $C^\bullet$ be a complex of $A[\phi]$-modules and suppose that each C^i has a slope $\leq h$-decomposition. Then every $H^i(C^\bullet)$ has a slope $\leq h$-decomposition, explicitly given by $H^i(C^\bullet) = H^i(C^\bullet_{\leq h}) \oplus H^i(C^\bullet_{>h})$.*

Proof. Clearly, we have $f(M_h) \subseteq N_h$. As M_h and N_h are finitely generated $A[\phi]$-modules, there exists a multiplicative polynomial $Q \in A[X]$ of slope $\leq h$ such that $Q^*(\phi)$ annihilates both M_h and N_h. Let $m' \in M^h$. Choose $m_1 \in M^h$ such that $Q^*(\phi)(m_1) = m'$, and write $f(m_1) = n + n'$, $\quad n \in N_h$, $n' \in N^h$. Then $f(m') = f\big(Q^*(\phi)(m_1)\big) = Q^*(\phi)(n) + Q^*(\phi)(n') = Q^*(\phi)(n') \in N^h$, which proves (1) in the case $M = M_h \oplus M^h$. This implies that the slope $\leq h$-decomposition of M is unique. To show $M_h = M_{\leq h}$, it is enough to prove that $M_{\leq h} \cap M^h = \{0\}$. For any $x \in M_{\leq h} \cap M^h$, there exists a multiplicative polynomial $T(X)$ of slope $\leq h$ such that $T^*(\phi)(x) = 0$. But $T^*(\phi) : M^h \to M^h$ is an isomorphism, which implies $x = 0$. Thus (1) is also complete, and (2) follows from (1). □

Definition 2.16. Let A be a Banach-Tate ring with a fixed multiplicative pseudo-uniformizer ϖ, and let M be a Banach A-module. Assume that M has a slope $\leq h$-decomposition $M = M_{\leq h} \oplus M_{>h}$. If $f : A \to B$ is a bounded morphism of Banach-Tate rings such that $f(\varpi)$ is a multiplicative pseudo-uniformizer in B, we say that the slope $\leq h$-decomposition is *functorial for f* if the decomposition $M \widehat{\otimes}_A B = (M_{\leq h} \otimes_A B) \oplus (M_{>h} \widehat{\otimes}_A B)$ is a slope $\leq h$-decomposition of $M \widehat{\otimes}_A B$ (using $f(\varpi)$ to define slopes for B). We say that the slope $\leq h$-decomposition is *functorial* if it is functorial for all such bounded homomorphisms of Banach-Tate rings out of A.

Theorem 2.17 ([JN19, Theorem 2.2.13])**.** *Let (A, A^+) be a Noetherian Tate-Huber pair with a fixed multiplicative pseudo-uniformizer ϖ, and let M be a Banach A-module with property* (Pr)*. Let ϕ be a compact A-linear operator on M, with Fredholm determinant F. If M has a slope $\leq h$-decomposition which is functorial with respect to $A \to \kappa(x)$ for all rank-1 points $x \in \mathrm{Spa}(A, A^+)$, then F has a slope $\leq h$-factorization. Conversely, if F has a slope $\leq h$-factorization, then M has a functorial slope $\leq h$-decomposition.*

Remark 2.18. Theorem 2.17 implies that a slope $\leq h$-decomposition of M is functorial if and only if it is functorial for the natural map $A \to \kappa(x)$ for all rank-1 points $x \in \mathrm{Spa}(A, A^+)$.

Moreover, combining this with the fact that the slopes of the Newton polygon of F correspond to the valuations of its zeros, we see that if M has a slope $\leq h$-decomposition and $m_x \in M_{\leq h} \otimes_A \kappa(x)$ is an eigenvector for ϕ with eigenvalue $\lambda_x \in \kappa(x)$, then λ_x has slope $\leq h$, i.e. $v_\varpi(\lambda_x) \leq h$. Here v_ϖ is the ϖ-adic valuation induced by ϖ, i.e. $v_\varpi(\varpi) = 1$.

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