Perfectoid spaces

Reference
- Berkeley lectures on p-adic geometry lectures 6, 7.
- Scholze's thesis
- Ben Heuer's note on perfectoid spaces
- étale cohomology of diamonds section 3.

Motivation

There are 2 kinds of non-Arch local fields

finite ext of $\mathbb{Q}_p$, $\mathbb{F}_p((t))$.

Nice if there're some direct relationship between them.

e.g. $\mathrm{Gal}_{\mathbb{Q}_p} \not\cong \mathrm{Gal}_{\mathbb{F}_p((t))}$, but

Thm (Fontaine – Winterberger thm)

$$\underbrace{\mathrm{Gal}_{\underbrace{\mathbb{Q}_p(p^{1/p^\infty})}_{\text{highly ramified over } \mathbb{Q}_p}}}_{\text{char } 0} \cong \underbrace{\mathrm{Gal}_{\mathbb{F}_p((t))}}_{\text{char } p}.$$

Perfectoid spaces allow you to generalize such things.

A better version:

$\mathrm{LHS} \cong \mathrm{Gal}_{\mathbb{Q}_p(p^{1/p^\infty})^\wedge}$ ($\mathcal{O}_{\mathbb{Q}_p(p^{1/p^\infty})}$ is Henselian)

$\mathrm{RHS} \cong \mathrm{Gal}_{\mathbb{F}_p((t))(t^{1/p^\infty})}$ (finite étale site is invariant under universal homeo. $\mathrm{Spec}\,\mathbb{F}_p((t)) \leftarrow \mathrm{Spec}\,\mathbb{F}_p((t))(t^{1/p^\infty})$ is a universal homeo)

$\cong \mathrm{Gal}_{\mathbb{F}_p((t))(t^{1/p^\infty})^\wedge}$ ($\mathcal{O}_{\mathbb{F}_p((t))(t^{1/p^\infty})}$ is Henselian)

then are both [illegible] fields

they are both perfectoid fields

Defn A perfectoid ring is a complete uniform Tate Huber ring R s.t. $\exists$ pseudouniformizer ϖ with $\underbrace{\varpi^p \mid p}_{\text{in } R^\circ}$ & $R^\circ/p \xrightarrow{x \mapsto x^p} R^\circ/p$ is surj.

Recall uniform : the set of powerbounded elts R° is bounded

Tate : $\exists$ topologically nilpotent unit

Rmk There is a related but different notion, called (integral) perfectoid rings. They are used in 'integral p-adic Hodge theory' by Bhatt – Morrow – Scholze & 'THH " " " " " " " '. But we won't need them. What we defined above is sometimes called perfectoid Tate ring.

Rmk Let R is a complete Huber ring, char $R = p$. Then R is perfectoid $\iff$ R is Tate & perfect.

$R \xrightarrow{(\cdot)^p} R$ is bijective

$\Longrightarrow$

$\Longleftarrow$ Banach open mapping theorem

Defn A perfectoid field K is a complete non-Arch ~~field~~ K s.t.

- K is not discretely valued
- $|p| < 1$
- $\underset{x \mapsto x^p}{\Phi}$ is surj on $\mathcal{O}_K/p$.

$|\cdot|$

$|xy| = |x||y|$

Rmk perfectoid field = perfectoid ring that is a field.

See 'On commutative nonarchimedean Banach fields' Kedlaya.
not obvious how to get $|\cdot|$

Examples of perfectoid rings

- $K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$ completion

$\mathbb{Q}_p(p^{1/p^\infty}) := \bigcup_{n\geq 0} \mathbb{Q}_p(p^{1/p^n})$

Pick compatible seq $p, p^{1/p}, p^{1/p^2}, p^{1/p^3}, \ldots$ s.t. $(p^{1/p^n})^p = p^{1/p^{n-1}}$.

complete ✓

uniform because $|\cdot|_p$ on $\widehat{\overline{\mathbb{Q}_p}}$ restricts to $\mathbb{Q}_p(p^{1/p^\infty})^\wedge$

roi of any non-Arch field is always bounded.

Tate: p is a pseudouniformizer

$(p^{1/p})^p \mid p$.

$$\mathcal{O}_K/p \xrightarrow{x \mapsto x^p} \mathcal{O}_K/p \quad ?$$

$$\| $$

$$\mathcal{O}_{\mathbb{Q}_p(p^{1/p^\infty})}/p$$

$(\mathcal{O}_{\hat{L}}/p = \mathcal{O}_L/p$ if p is a pseudouniformizer of L $)$

$$\| $$

$$\mathbb{Z}_p[p^{1/p^\infty}]/p$$

$$\cup$$

e.g. $a_1 p^{1/p} + a_2 p^{3/p^5}$

$$\uparrow \Phi$$

$a_1 p^{1/p^2} + a_2 p^{3/p^6}$.

General case is similar.

$\therefore$ $\mathbb{Q}_p(p^{1/p^\infty})^\wedge$ is perfectoid.

Example 2 $\mathbb{F}_p((t))(t^{1/p^\infty})^\wedge$ is perfectoid

$\mathbb{Q}_p(p^{1/p^\infty}) \overset{p \leftrightarrow t}{\longleftrightarrow} \bigcup_{n\geq 1} \mathbb{F}_p((t))(t^{1/p^n}) =: \mathbb{F}_p((t^{1/p^\infty}))$

'tilting'

Explicit description

$$\mathbb{F}_p((t^{1/p^\infty})) = \left\{ \sum_{i\in\mathbb{Z}[\frac{1}{p}]} a_i t^i \;\middle|\; \begin{array}{l}\forall n\in\mathbb{N}, \text{ there are only finitely} \\ \text{many } i\leq n \text{ with } a_i\neq 0\end{array} \right\}$$

Example 3 $\quad \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$ is perfectoid

Example 4 $\quad K\langle T^{1/p^\infty}\rangle$ perfectoid closed unit disc over K,

$\quad\quad \overset{\shortparallel}{\mathcal{O}_K[T^{1/p^\infty}]^\wedge[\frac{1}{\varpi}]}$ where K is a perfectoid field

ϖ pseudouniformizer of K

Explicit description

$$K\langle T^{1/p^\infty}\rangle = \left\{ \sum_{i\in\mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_i T^i \;\middle|\; \begin{array}{l}\forall n\in\mathbb{N}, \; a_i\in\varpi^n\mathcal{O}_K \\ \text{for a.e. } i\end{array} \right\}$$

Example 5 $\quad \mathbb{Q}_p(\mu_{p^\infty})^\wedge$ is perfectoid

Tilting equivalence

R perfectoid ring

Defn $\quad R^\flat := \varprojlim_{\Phi} R = \varprojlim \left(R \xleftarrow{\Phi} R \xleftarrow{\Phi} R \leftarrow \cdots\right)$, $\quad \Phi = (-)^p$

$$= \{(x_i)\in \prod_{i\in\mathbb{Z}_{\geq 0}} R \;:\; x_{i+1}^p = x_i \quad \forall i\geq 0\}.$$

addition on $R^\flat$

$$(x_i)+(y_i) = \lim_{m\to\infty} (x_{i+m}+y_{i+m})^{p^m}$$

multiplication on $R^\flat$

$$(x_i)\cdot(y_i) = (x_i y_i)$$

<u>topology</u>
inverse limit topology

<u>Fact</u> • $R^\flat$ is a perfectoid ring of char p.

• $(R^\flat)^\circ = (R^\circ)^\flat := \varprojlim_{\Phi} R^\circ \xrightarrow{\sim} \varprojlim_{\Phi} (R^\circ/\varpi)$.
any pseudo-uniformizer $\varpi | p$.

• $\exists$ pseudo-uniformizer $\varpi | p$ which admits a seq of compatible p-power roots ϖ^{1/p^n} in R. Write
$$\varpi^\flat = (\varpi, \varpi^{1/p}, \varpi^{1/p^2}, \dots) \in R^\flat.$$
Then
$$R^\circ/\varpi \cong R^{\circ\flat}/\varpi^\flat \quad \& \quad R^\flat = R^{\circ\flat}\left[\tfrac{1}{\varpi^\flat}\right].$$

<u>Thm</u> (Tilting equiv)
Let R be a perfectoid ring. Then
$$\underbrace{R\text{-Perf}}_{\text{cat of perfectoid alg over } R} \xrightarrow[\sim]{(\ )^\flat} R^\flat\text{-Perf}.$$
It induces
$$\underbrace{R_{\text{fét}}}_{\text{cat of finite étale } R\text{-algs}} \xrightarrow[\sim]{(\ )^\flat} R^\flat_{\text{fét}}.$$

<u>Pf sketch</u> (Scholze's thesis, Ben Heuer's note Thm 1.2.37):
Assume $R=K$ perfectoid field. We'll use almost mathematics.
(See Scholze's thesis)

$$K\text{-Perf} \simeq \underbrace{\mathcal{O}_K^a\text{-Perf}}_{\text{cat of almost perfectoid } \mathcal{O}_K\text{-alg}} \simeq (\mathcal{O}_K/\varpi)^a\text{-Perf}$$

Proved by showing all relevant $\underbrace{\text{contangent complexes}}_{\text{generalization of Kähler differentials}}$ $=0$.

$S \mapsto S^{\circ a}$
$A[\frac{1}{\varpi}] \leftarrow A$

$A \mapsto A/\varpi$

E.g. If R is a perfect $\mathbb{F}_p$-alg, then $\Omega^1_{R/\mathbb{F}_p} = 0$. Indeed,
$dr = d(r^{1/p})^p = p(r^{1/p})^{p-1} d(r^{1/p})$

Similarly,

$K^\flat\text{-Perf} \simeq \mathcal{O}_{K^\flat}^a\text{-Perf} \simeq (\mathcal{O}_{K^\flat}/\varpi^\flat)^a\text{-Perf}$ $=0$

But $\mathcal{O}_K/\varpi = \mathcal{O}_{K^\flat}/\varpi^\flat$, so get this equality.

$\therefore$ $K\text{-Perf} \simeq K^\flat\text{-Perf}$.

Can prove $K_{fét} \simeq K^\flat_{fét}$ similarly, but need one extra ingredient, called almost purity thm below. $\square$

Almost purity thm Let R be a perfectoid ring $R \to S$ be a finite étale map. Then S is perfectoid & $\underbrace{R^\circ \to S^\circ}$ is almost finite étale.

not finite in general,
Only almost finite.

Perfectoid spaces

Defn A perfectoid space is an adic space X which admits an open cover by $\mathrm{Spa}(R,R^+)$, where R is perfectoid.

Examples

- $\mathrm{Spa}\ \mathbb{Q}_p(p^{1/p^\infty})^\wedge$

- Let K be a perfectoid ring.
  If S is a profinite set, then
$$\underline{S} \times \mathrm{Spa} K := \varprojlim_i (\underbrace{S_i \times \mathrm{Spa} K}_{\text{disjoint union of Spa} K}) \quad \text{where} \quad S = \varprojlim_i \underbrace{S_i}_{\text{finite}}$$
$$= \mathrm{Spa}\ \underbrace{\mathrm{Cont}}_{\text{continuous functions}}(S, K)$$
  is a perfectoid space. This gives a fully faithful embedding
$$(\text{profinite set}) \hookrightarrow (\text{perfectoid spaces over } \mathrm{Spa} K).$$

- any open subspace of a perfectoid space is perfectoid (by the fact below)

Fact Let R be a perfectoid ring, $X = \mathrm{Spa}(R, R^+)$.
Then $\forall$ rational open subset $U \subset X$,

$$\mathcal{O}_X(U) \text{ is } \underbrace{\text{perfectoid}}_{\Rightarrow \text{ uniform}}.$$

$\Rightarrow$ X is stably uniform (i.e. $\forall$ rational open $U \subset X$, $\mathcal{O}_X(U)$ is rational.)
$\Rightarrow$ (R, R^+) is sheafy.

Warning It is unknown whether a perfectoid space which is affinoid must be of the form $\mathrm{Spa}(R, R^+)$ for a perfectoid ring R.
(being perfectoid only implies there is a cover by such $\mathrm{Spa}(R, R^+)$. We don't know whether the cover can be taken to be a singleton)
By an affinoid perfectoid, we & most people mean $\mathrm{Spa}(R, R^+)$ for a perfectoid ring R.

Tilting

By gluing, we can generalize tilting to perfectoid spaces to get a functor

$$\begin{aligned}(\text{perfectoid spaces}) &\longrightarrow (\text{perfectoid spaces}) \\ X &\longmapsto X^\flat\end{aligned}$$

Thm $\cdot$ $|X| \cong |X^\flat|$
underlying top space

$\cdot$ (Tilting equiv) $\mathrm{Perfd}_X \overset{(\ )^\flat}{\simeq} \mathrm{Perfd}_{X^\flat}$
& $X_{\text{ét}} \simeq X^\flat_{\text{ét}}$
& $X_{\text{fét}} \simeq X^\flat_{\text{fét}}$.

great thm because it allows to relate char 0 & char p things.

Example $X = \mathrm{Spa}\, \mathbb{Q}_p(p^{1/p^\infty})^\wedge$. Then $X^\flat = \mathrm{Spa}\, \mathbb{F}_p((p^{1/p^\infty}))$.

$\Rightarrow X_{\text{fét}} \simeq X^\flat_{\text{fét}}$.

$\xRightarrow{\pi_1^{\text{ét}}}$ Gal [illegible] $\simeq$ Gal [illegible] (can already deduced)

$\mathrm{Gal}(\overline{\mathbb{Q}_p(p^{1/p^\infty})}) \cong \mathrm{Gal}(\overline{\mathbb{F}_p((p^{1/p^\infty}))})$. (from $K_{\text{fét}} \simeq K^\flat_{\text{fét}}$)

Thm (Almost acyclicity) ← this is one nice property of perfectoid spaces.

$\forall$ affinoid perfectoid space X,

$$H^i(X, \mathcal{O}_X^+) \overset{a}{=} 0 \quad \forall i \geq 1.$$

almost

$$H^i_{\text{ét}}(X, \mathcal{O}_X^+) \overset{a}{=} 0 \quad \forall i \geq 1.$$

almost

See 'étale cohom of diamonds' 3.24, 6.3.

Rmk This is false for general adic spaces.
False even for rigid analytic spaces
e.g. $H^2_{\text{ét}}(\mathrm{Spa}\, \mathbb{C}_p\langle T\rangle, \mathcal{O}^+)$ is huge.

App Hodge-Tate comparison.

Thm X sm proper rigid space / $\mathbb{C}_p$.
Have a spectral seq $E_2^{ij} = H^i(X, \Omega_X^j(-j)) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{C}_p)$.
It degenerates at E_2-page.

(Classically, if X cpt Kähler mfd, then $\bigoplus_{i+j=n} H^i(X, \Omega_X^j) = H^n_{\text{sing}}(X, \mathbb{C})$
We do not put any condition analogous to 'Kähler' here.)

Pf idea
Realize the HT ss as the Leray ss for the mor of sites
$$\nu : X_{\text{proét}} \longrightarrow X_{\text{ét}}$$
& the sheaf $\widehat{\mathcal{O}}_X$.

Leray ss : $$H^i(X, \underbrace{R^j\nu_*\widehat{\mathcal{O}}_X}_{\cong\, \Omega_X^j(-j)}) \Rightarrow H^{i+j}_{\text{proét}}(X, \widehat{\mathcal{O}}_X)$$

$$H^{i+j}_{\text{proét}}(X, \widehat{\mathcal{O}}_X) \cong H^{i+j}_{\text{ét}}(X, \mathbb{C}_p)$$

called primitive comparison thm

use perfectoid cover & almost purity

use tilting argument to relate it to the Artin-Schreier seq. $\square$

More details in Ben Heuer's note on perfectoid spaces.

Rmk

If char $R = p$, then $R^\flat = R$.

$$\begin{array}{rcl} R^\flat = \varprojlim_{\Phi} R & \longleftarrow & R \\ (r^{1/p^n})_{n \geq 0} & \longleftarrow\!\shortmid & r \\ (s_i) & \longmapsto & s_0 \end{array}$$

because char p perfectoid rings are perfect