Adic Spaces

1) Recap: Huber pairs + affinoid adic spaces

2) definition of an adic space (structure sheaf, local rings) + properties
! mirrored in scheme theory

3) Morphisms of adic spaces ("adic" morphisms, finite type/proper/étale/smooth ...)

4) a quick survey:

adic spaces (mimicking schemes) vs formal schemes (models) vs rigid analytic spaces (mimicking varieties to schemes; max-spec?) vs Berkovich spaces (Hausdorff + arc connected (more top.))

essentially all try to mimic {schemes over $\mathbb{C}$} $\xrightarrow{\text{analytification}}$ {complex analytic spaces}
GAGA

1) **Recap**: Let k be a

Def: A Huber ring is a topological ring A with

- $A_0 \subseteq A$ open subring ("ring of definition")
- $I \subseteq A_0$ ideal ("ideal of definition")

such that

- the topology on A_0 is the I-adic topology
- the ideal I is finitely generated

Examples: $\mathbb{Q}_p$, $\mathbb{Z}_p[[w]]$ ((p,w)-adic top.), $K\langle T\rangle$ ("Tate ring") ((ϖ)-adic top.)

A°: power bounded elements
$A^{\circ\circ}$: topologically nilpotent elements

Def.: A Huber pair (A, A^+) is a Huber ring A with a subring $A^+ \subseteq A^\circ$, open and integrally closed in A.

later $A^+ = A^\circ$ often

Examples: $(\mathbb{Q}_p, \mathbb{Z}_p)$, $(\mathbb{Z}_p[[w]], \mathbb{Z}_p[[w]])$, $(K\langle T\rangle, K^\circ\langle T\rangle)$

Def: Adic spectrum

$$\mathrm{Spa}(A, A^+) = \{x \in \underbrace{\mathrm{Cont}(A)}_{\text{cont. valuations}} \mid |f(x)| \leq 1 \ \forall f \in A^+\}$$

Support map:

$$\mathrm{supp}: \mathrm{Spv}(A)\ (:= \mathrm{Cont}(A_{disc})) \longrightarrow \mathrm{Spec}\, A$$
$$x \longmapsto \{f \in A \mid |x(f)| = 0\}$$

Important

Example: Closed unit disk (p-adic) $\mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T\rangle) = \{x \in \mathrm{Cont}(\mathbb{C}_p\langle T\rangle) \mid |T(x)| \leq 1\}$

type		name	supp(x)	T_x	A_x/m_x	closed?
1	classical	x_α	$\langle T-\alpha\rangle$	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}}_p$	closed
2	Gauss points →	$x_{\alpha,r}$ $(r \in p^{\mathbb{Q}})$	$\{0\}$	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}}_p(t)$	non-closed
3		$x_{\alpha,r}$ $(r \notin p^{\mathbb{Q}})$	$\{0\}$	$p^{\mathbb{Q}} r^{\mathbb{Z}} \subseteq \mathbb{R}^\times$	$\overline{\mathbb{F}}_p$	closed
4		$x_{D_\bullet}$ $(\cap D_i = \emptyset)$	$\{0\}$	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}}_p$	closed
5		$x^\lambda_{\alpha,r}$ $(r \in p^{\mathbb{Q}})$	$\{0\}$	$p^{\mathbb{Q}} \times (\frac{1}{2})^{\mathbb{Z}}$	$\overline{\mathbb{F}}_p$	closed

in closure

Def.: A <u>rational subset</u> is of the form

$$U\left(\frac{f_1,\dots,f_n}{g}\right) = \{x \in \mathrm{Spa}(A,A^+) \mid |f_i(x)| \leq |g(x)| \neq 0\}$$

if $(f_1,\dots,f_n,g)$ generate an open ideal.

<u>Structure (pre) sheaf</u>:

$\mathcal{O}_X(X) = \hat{A}$, $\mathcal{O}_X^+(X) = \hat{A}^+$

on rational subsets $U\left(\frac{f_1,\dots,f_n}{g}\right)$ we have

$A_U :=$ completion of $A[\frac{1}{g}]$

$A_U^+ :=$ completion of $A^+\left[\frac{f_1}{g},\dots,\frac{f_n}{g}\right]^N$ ← integral closure in A_g

$(A, A^+) \longrightarrow (A_U, A_U^+)$

$\mathrm{Spa}(A_U, A_U^+) \hookrightarrow \mathrm{Spa}(A, A^+)$

A_U

$\left(A_0\left[\frac{f_1,\dots,f_n}{g}\right], I A_0\left[\frac{f_1,\dots,f_n}{g}\right]\right)$ of definition

so $\mathcal{O}_X(U) = A_U$, $\mathcal{O}_X^+(U) = A_U^+$

Warning: not always a sheaf

↳ strongly noetherian (all $A\langle T_1,\dots\rangle$ noeth.) or perfectoid ($\Rightarrow$ stably uniform) (rational subsets $\mathcal{O}_X(U)$ uniform, A° bounded)

Think schemes → sheafy

2) <u>Definition of adic spaces</u> ⇝ glue together affinoids

Def. A locally v-ringed space $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$ is a triple where
- $(X, \mathcal{O}_X)$ locally topologically ringed space
- for each $x \in X$ equ. class of valuations v_x on local ring $\mathcal{O}_{X,x}$ who's support is the maximal ideal

Morphisms of locally v-ringed spaces

$$(X, \mathcal{O}_X, \{v_x\}_{x\in X}) \longrightarrow (Y, \mathcal{O}_Y, \{v_y\}_{y\in Y})$$

are morphisms of locally ringed spaces s.t. for $x \mapsto y$

$v_x|_{\mathcal{O}_{Y,y}} = v_y$ $\quad \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$

open immersion if X homeomorphism onto open $U \subseteq Y$

and it induces an isomorphism

$$(X, \mathcal{O}_X, \{v_x\}_{x\in X}) \xrightarrow{\sim} (U, \mathcal{O}_Y|_U, \{v_u\}_{u\in U})$$

sheafy Huber pair (A, A^+) $\longrightarrow$ $(X, \mathcal{O}_X, \{v_x\}_{x\in X})$

(A, A^+) $\rightsquigarrow$ $X = \mathrm{Spa}(A, A^+)$
$\mathcal{O}_X = \mathcal{O}_X$
v_x = limit of valuation determined by x on $\mathcal{O}_X(U)$ (U rational)

<u>Rmk</u>: $\{v_x\}_{x\in X}$ determines a subsheaf of $\mathcal{O}_X$ given by $\mathcal{O}_X^+(U) = \{ a \in \mathcal{O}_X(U) \mid v_x(a_x) \leq 1 \text{ for all } x \in U\}$

<u>Def</u>: A locally v-ringed space obtained from a Huber pair is call affinoid adic space.

An <u>adic space</u> is a locally v-ringed space that admits an open covering $X = \bigcup_i U_i$ by affinoid adic spaces.

Just like in the scheme case we can glue affinoid adic spaces to adic spaces.

<u>Example</u>: The affine line

k non-archimedean field with pseudo-uniformizer $\varpi \in k$

closed disk of radius $|\varpi^{-n}|$

$$\mathbb{D}_k(0, |\varpi^{-n}|) = \mathrm{Spa}(k\langle \varpi^n T\rangle, k^\circ\langle \varpi^n T\rangle)$$

indeed for a classical point x_α corresponding to $\alpha \in k$ via $x_\alpha(f) = |f(\alpha)|$ we have

$$x_\alpha \in \mathbb{D}_k(0, |\varpi^{-n}|) \iff |\varpi^n T(x_\alpha)| \leq 1$$

i.e. $|\varpi^n \alpha| \leq 1$
i.e. $|\alpha| \leq |\varpi^{-n}|$

Inclusions: $m \geq n$ $\quad (k\langle \varpi^m T\rangle, k^\circ\langle \varpi^m T\rangle) \hookrightarrow (k\langle \varpi^n T\rangle, k^\circ\langle \varpi^n T\rangle)$

$\rightsquigarrow \mathbb{D}_k(0, |\varpi^{-n}|) \hookrightarrow \mathbb{D}_k(0, |\varpi^{-m}|)$

affine $\mathbb{A}^1_k = \bigcup_n \mathbb{D}_k(0, |\varpi^{-n}|)$ (name: analytification of the affine line)

NOT affinoid:

$\rightsquigarrow \bigcup_n \mathbb{D}_k(0, |\varpi^{-n}|)$ has no finite subcovering

but affinoid adic spaces are quasi-compact

Global sections of structure sheaf:

$\mathcal{O}_{\mathbb{A}^1_k}(\mathbb{A}^1_k) = k\{\{T\}\}$ power series that converge everywhere

ex. $\sum_{i=0}^{\infty} \varpi^{i^2} T^i \in k\{\{T\}\}$

other examples

open unit disk $\mathring{\mathbb{D}}_k(0,1) = \bigcup_{m \in \mathbb{N}} \mathbb{D}_k(0, |\varpi^{1/m}|) \neq$ Caution $\{x \in \mathbb{D}_k(0,1) \mid |T(x)| < 1\}$

global sections

subring of $k[[T]]$ s.t. converge for all $a \in k$ of norm < 1

projective line: glue two unit disks

$\mathbb{D}_1 := \mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$ $\quad \mathbb{D}_2 := \mathrm{Spa}(k\langle S\rangle, k^\circ\langle S\rangle)$

along $\{|T| = 1\}$, $\{|S| = 1\}$ via

$$(k\langle S, S^{-1}\rangle, k^\circ\langle S, S^{-1}\rangle) \xrightarrow[S \mapsto T^{-1}]{\sim} (k\langle T, T^{-1}\rangle, k^\circ\langle T, T^{-1}\rangle)$$

Indeed $\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) = k$

<u>Local Huber pairs</u>:

Goal: define $(\mathcal{O}_{X,x}, \mathcal{O}^+_{X,x})$ as Huber pair i.e. structure of affinoid adic space

<u>Def.</u>: A Huber pair (A, A^+) is called local if

- A, A^+ are local
- A^+ is the preimage of a valuation ring k_A^+ of the residue field k_A of A
- the valuation v_A of A defined by k_A^+ is continuous

$$\begin{array}{ccc} A & \twoheadrightarrow & A/m = k_A \\ \cup\!| & & \cup\!| \\ A^+ & \twoheadrightarrow & k_A^+ \end{array}$$
↗ valuation ring

One can show $m \subseteq A^+$ and $A^+/m = k_A^+$

Example: $(\mathbb{Q}_p, \mathbb{Z}_p)$

$k_A = \mathbb{Q}_p \qquad k_A^+ = \mathbb{Z}_p$

k_A^+ defines p-adic valuation

$(\mathbb{Z}_p, \mathbb{Z}_p)$

We have $k_A = \mathbb{F}_p$, $k_A^+ = \mathbb{F}_p$

k_A^+ defines trivial valuation on k_A

Lemma: Let (A, A^+) be a local Huber pair, $\mathfrak{a} \subseteq A^+$ an ideal. Then either $\mathfrak{a} \subseteq m$ or $m \subseteq \mathfrak{a}$.

Proof: If $\mathfrak{a} \not\subseteq m$, take $m \in m$, and $a \in \mathfrak{a} \setminus m$.

$\rightsquigarrow$ thus a invertible in A so $a^{-1} \in A$, $a^{-1}m \in m \subseteq A^+$

$\rightsquigarrow$ $m = a a^{-1} m \in \mathfrak{a} \Rightarrow m \subseteq \mathfrak{a}$.

So let $I \subseteq A^+$ be an ideal of definition.

$\rightsquigarrow$ $I \subseteq m$ or $m \subseteq I$

Def: (i) $I \subseteq m$ "formal case"

(ii) $I = (0) \subseteq m$ "discrete case" $\rightsquigarrow (\mathbb{Z}_p, \mathbb{Z}_p)$ discrete val on k_A

(iii) $m \subseteq I$ "analytic case" $\rightsquigarrow (\mathbb{Q}_p, \mathbb{Z}_p)$ p-adic val. on k_A

Let X be an adic space, $x \in X$, $\mathrm{Spa}(B, B^+)$ an affinoid nbd. (B_0, J) ring of definition

Idea: local Huber pair $(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}^+)$ via colimit

But colimit topology $\neq$ valuation topology defined by x

$\rightsquigarrow$ "define new colimit by weakening condition"

Construction: Let $\mathfrak{p} := \operatorname{supp} x \subseteq B$

Define $A := B_{\mathfrak{p}}$ with $\pi : B_{\mathfrak{p}} \twoheadrightarrow k(\mathfrak{p})$ ↪ x defines valuation

set $A^+ := \{a \in A \mid \pi(a) \in k(x)^+\}$

set $I = JA^+ \subseteq A^+$ as ideal of definition

Prop.: For two affinoid opens (B, B^+), (B', B'^+)

we have $(\hat{A}, \hat{A}^+) \xrightarrow{\sim} (\hat{A}', \hat{A}'^+)$

Prop: The pair constructed above satisfies

$$X_x := \operatorname{Spa}(A, A^+) \sim \lim_{x \in U} U$$

We will use

$$X_x = \operatorname{Spa}(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}^+)$$

equipped with not the colimit topology, but

$(\mathcal{O}_{X,x}^+, I\mathcal{O}_{X,x}^+)$ as ring of definition.

After completion we have $(\widehat{\mathcal{O}_{X,x}}, \widehat{\mathcal{O}_{X,x}^+}) \xrightarrow{\sim} (\hat{A}, \hat{A}^+)$
with this topology

Examples: For affinoid Huber pairs $(\mathbb{Z}_p, \mathbb{Z}_p)$

$\operatorname{Spec} \mathbb{Z}_p$: (0), (p)

residue field $\mathbb{Z}_{p(0)} = \mathbb{Q}_p$ ⇝ only p-adic valuation

analytic so $x :=$ p-adic valuation on $\mathbb{Z}_p$ w local Huber pair $(\mathbb{Q}_p, \mathbb{Z}_p)$

residue field $\mathbb{Z}_p/(p) = \mathbb{F}_p$ ⇝ only trivial valuation

not analytic so x with $p(x) = 0$, $a(x) = 1$ w. local Huber pair $(\mathbb{Z}_p, \mathbb{Z}_p)$

Analytic adic spaces:

Def.: Let (A, A^+) be a Huber pair.

$x \in \operatorname{Spa}(A, A^+)$ is called analytic if $\operatorname{supp} x$ is not open ⇝ i.e. small

Let X be an adic space.

$x \in X$ is called analytic, if it has affinoid nbd $\operatorname{Spa}(A, A^+)$ s.t. $x \in \operatorname{Spa}(A, A^+)$ is analytic

An adic space X is called analytic if all points are analytic

→ later, rigid analytic spaces

<u>Lemma</u>: A morphism $f: \mathrm{Spa}(A,A^+) \longrightarrow \mathrm{Spa}(B,B^+)$ of adic spaces non-analytic points to non-analytic points.

<u>Proof</u>: f is given by $\varphi: (B,B^+) \longrightarrow (A,A^+)$

If $\mathrm{supp}(x) \subseteq A$ is open then so is $\varphi^{-1}(\mathrm{supp}(x)) = \mathrm{supp}(f(x))$.

3) <u>Morphisms of adic spaces</u>:

Recall that a morphism of adic spaces is a morphism

$$(X, \mathcal{O}_X, \{v_x\}_{x\in X}) \longrightarrow (Y, \mathcal{O}_Y, \{v_y\}_{y\in Y})$$

where $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces

and $v_x|_{\mathcal{O}_{Y,y}} = v_y$ on $\mathcal{O}_{X,x}$.

For affinoid it is induced by a morphism of Huber pairs $(A, A^+) \longrightarrow (B, B^+)$

exist rings of def. s.t. $\varphi(A_0) \subseteq B_0$, $\varphi(A^+) \subseteq B^+$

<u>Def.</u>: A homomorphism of Huber rings $\varphi: R \longrightarrow S$ is called <u>adic</u> if there exists rings of definition R_0, S_0 s.t. $\varphi(R_0) \subseteq S_0$ and for an ideal of def., $\varphi(I)S_0$ is an ideal of def.

A morphism of adic spaces is <u>adic</u> if it is locally defined by adic morphisms of Huber pairs.

<u>Rmk</u>: topology of S is entirely determined by R!

<u>Lemma</u>: An adic morphism takes analytic points to analytic points.

Proof see HLW

Finiteness:

we cannot take fin. gen. $\rightsquigarrow$ $k\langle T\rangle$ is not

Def. A homo. $R \to S$ of complete Huber rings is a quotient map if it is surj., continuous, open.
A homo $(R,R^+) \to (S,S^+)$ is a quotient map if $R \to S$ is a quotient map, S^+ is the relative integral closure of the image of R^+ in S.

Def: A homomorphism $R \to S$ of Huber rings is strictly of topologically finite type if there is a quotient map $R\langle T_1,\dots,T_n\rangle \twoheadrightarrow \hat{S}$ of $\hat{R}$-algebras

For definition of "finite type" see

HLV4

Further properties:

Fiber products $X \times_Z Y$ exist in "nice" cases

→ perfectoid spaces
→ f,g finite type
→ g adic, f weakly finite type

Can define separated/proper via

Def: f quasisep., locally of finite type

$$\begin{tikzcd}
\mathrm{Spa}(K,K^\circ) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spa}(K,K^+) \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

K non arch field, $K^+ \subseteq K^\circ$ open valuation subring

flat, smooth, étale via lifting

<u>Def.</u>: Let $f: X \longrightarrow Y$ be a morphism of locally finite type, locally noeth. adic space

(A, A^+) Noetherian Huber pair $I \subseteq A$, $I^2 = 0$

$S = \mathrm{Spa}(A, A^+)$, $T = \mathrm{Spa}(A/I, (A/I)^+)$

$k[\varepsilon]/(\varepsilon^2)$

for any such A,

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

smooth lifts

étale unique

4) Comparison:

Analytification: For $(Z, \mathcal{O}_Z, \{v_z\}_{z\in Z})$ write $\underline{Z} = (Z, \mathcal{O}_Z)$ locally ringed space

Fiber product with schemes

Let $Y \longrightarrow X$ be a morphism of schemes that is locally of finite type, Z a stable adic space, $\underline{Z} \longrightarrow X$ morphism of locally ringed spaces. Then the fiber product $Y \times_X Z$ exists, is a stable adic space, and the projection

$$Y \times_X Z \longrightarrow Z$$

is locally of finite type.

$$\begin{tikzcd} W \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \underline{Y \times_X Z} \arrow[r] \arrow[d] & \underline{Z} \arrow[d] \\ & Y \arrow[r] & X \end{tikzcd}$$

Def: Let (k, k^+) be an affinoid field.

The analytification of a variety X/k is defined as

$$X^{an} := X \times_{\operatorname{Spec} k} \operatorname{Spa}(k, k^+)$$

construction amounts to gluing $k\langle \varpi^k T_1, \dots, \varpi^k T_n\rangle / I$ if $X = k[T_1, \dots, T_n]/I$

$\uparrow$ bounded

Example: affine line $\mathbb{A}^1_k$ (variety!)

give affinoids $\operatorname{Spa}(k\langle \varpi^n T\rangle, k^\circ\langle \varpi^n T\rangle)$

via inclusion

$$\Rightarrow (\mathbb{A}^1_k)^{an} = \bigcup_n \operatorname{Spa}(k\langle \varpi^n T\rangle, k^\circ\langle \varpi^n T\rangle)$$

$$= \mathbb{A}^1_{(k, k^\circ)} \quad \text{adic affine line}$$

$$\begin{tikzcd} \underline{\mathbb{A}^{1\,an}_k} \arrow[r] \arrow[d] & \underline{\operatorname{Spa}(k, k^\circ)} \arrow[d] \\ \mathbb{A}^1_k \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Formal schemes:

A noetherian ring endowed with I-adic topology

formal spectrum $\mathfrak{X} = \mathrm{Spf}\ A$

- underlying topological space is all open prime ideals (i.e. $\mathrm{Spec}\ A/I$)
- structure sheaf $\mathcal{O}_{\mathfrak{X}}(\mathrm{Spf}\ A_f) = \hat{A}_f$

"infinitesimal thickening" of a standard scheme

→ glue to locally noetherian formal schemes

Thm: There is a fully faithful functor from locally noetherian formal schemes into category of adic spaces. It maps

$$A \longmapsto \mathrm{Spa}(A, A)$$

Given a locally noetherian formal scheme, we can construct

$$(X, \mathcal{O}_X) \xrightarrow{i_X} (X^{ad}, \mathcal{O}_{X^{ad}}, \mathcal{O}^+_{X^{ad}}) \xrightarrow{r_X} (X, \mathcal{O}_X)$$

$\mathcal{O}_{X^{ad}} = \mathcal{O}^+_{X^{ad}}$

i_X : maps open prime ideal $\mathfrak{p}$ to trivial valuation on $k(\mathfrak{p})$ (i.e. a point $x \in X^{ad}$ w. $\mathrm{supp}\ x = \mathfrak{p}$) ⇝ induces trivial val

r_X: $x \in X^{ad}$, we have

$k(x)^+$ valuation ring in $k(x)$

and $A \longrightarrow k(x)$

↪ image in $k(x)^+$ since $|a(x)| \leq 1\ \forall a \in A$

therefore we have a map $\varphi: \mathrm{Spec}\ k(x)^+ \longrightarrow \mathrm{Spec}\ A$

so we send x to $\varphi(\mathfrak{m})$ where $\mathfrak{m} \in k(x)^+$ unique closed point

→ $r_X \circ i_X = \mathrm{id}$

Rigid analytic varieties:

Def: Let (k, k°) be an analytic field (field with valuation topology)

A *rigid analytic variety* over (k, k°) is an adic space X of finite type over (k, k°).

original theory

Def.: A *formal model* of a rigid analytic variety over (k, k°) is a formal scheme $\mathfrak{X}$ over $\mathrm{Spa}(k^\circ, k^\circ)$ with $\mathfrak{X}_\eta = X$.

$\uparrow$ two points generic $\rightsquigarrow$ special

$(\mathbb{Z}_p, \mathbb{Z}_p)$ p-adic trivial on $\mathbb{F}_p$

Given a formal model $\mathfrak{X}$, blowing up along a sheaf of ideals $\mathcal{J}$, s.t. $\varpi \in \mathcal{J}$ we obtain another formal model.

Thm (Raynaud)

There is an equivalence of categories

$$\left\{\begin{matrix}\text{qc } \varpi\text{-torsion free formal } k^\circ\text{-schemes} \\ \text{of topologically finite type}\end{matrix}\right\}$$

localised by admissible formal blowups

$$\xrightarrow{\sim} \{\text{qcqs rigid } k\text{-spaces}\}$$

$$\mathfrak{X} \longmapsto \mathfrak{X}_\eta \quad (\text{generic fiber})$$

In particular, for qcqs rigid k-spaces, formal models always exist.

Further, given a qcqs rigid k-space X and a formal model $\mathfrak{X}_i$ we have a specialisation map

$$sp_i : X \hookrightarrow \mathfrak{X}_i \qquad (X \hookrightarrow \mathfrak{X}^{ad} \xrightarrow{r_X} \mathfrak{X})$$

Thm: $sp = (sp_i)_{i \in I} : X \longrightarrow \varprojlim_i \mathfrak{X}_i$ is a homeomorphism and induces an iso

$$(X, \mathcal{O}_X^+) \cong \varprojlim_i (\mathfrak{X}_i, \mathcal{O}_{\mathfrak{X}_i})$$

Berkovich spaces:

Glued locally from affinoid algebras A (quotients of Tate alg.)

$$\mathcal{M}(A) = \{ \text{multiplicative seminorms } v : A \longrightarrow \mathbb{R}_{\geq 0} \}$$

topology: weakest top. s.t. $x \longmapsto |x(f)|$ is continuous $\forall f \in A$

Given an adic space X

the separated quotient X^B is a Berkovich space.

$\rightsquigarrow$ separated quotient is the largest quotient s.t. X is Hausdorff.

In particular, we identify every point to its maximal generization

$\rightsquigarrow$

Closed unit disk (p-adic) $\mathrm{Spa}(\mathbb{C}_p\langle T\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T\rangle) = \{ x \in \mathrm{Cont}(\mathbb{C}_p\langle T\rangle) \mid |T(x)| \leq 1 \}$

in closure

classical

Gauss points

type	name	supp(x)	T_x	A_x/m_x	closed?
1	x_α	$\langle T-\alpha\rangle$	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}}_p$	closed
2	$x_{\alpha,r}$ $(r \in p^{\mathbb{Q}})$	$\{0\}$	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}}_p(t)$	non-closed
3	$x_{\alpha,r}$ $(r \notin p^{\mathbb{Q}})$	$\{0\}$	$p^{\mathbb{Q}} r^{\mathbb{Z}} \subseteq \mathbb{R}^\times$	$\overline{\mathbb{F}}_p$	closed
4	$x_{D_\bullet}$ $(\cap D_i = \emptyset)$	$\{0\}$	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}}_p$	closed
5	$x_{\alpha,r}^{\lambda}$ $(r \in p^{\mathbb{Q}})$	$\{0\}$	$p^{\mathbb{Q}} \times (\tfrac{1}{2})^{\mathbb{Z}}$	$\overline{\mathbb{F}}_p$	closed

identified

$\rightsquigarrow$ only rank 1 points left