Recall : Tate ring and affinoid alg.

k non-arch., R val ring of k,

Def. n-variable Tate ring T_n / k is

$$T_n(k) = \left\{ \sum_{J \in \mathbb{Z}^n_{\geq 0}} a_J X^J \in k[[X_1, \cdots, X_n]] : |a_J| \to 0,\ J \to \infty \right\}.$$
$$\| \quad k\langle X_1, \cdots, X_n \rangle$$

The Gauss norm on T_n is

$$\left\| \sum_J a_J X^J \right\| = \max_J |a_J|$$

$(\|f\| = 0 \iff f = 0)$

Easy to check Gauss norm is a non-arch val on k-Banach alg $T_n(k)$.

We can evaluate $f \in T_n(k)$ at closed unit cycle of $\mathbb{Q}_p$ ($\mathbb{C}_p$), which will give us a model of closed unit ball of $\mathbb{C}_p$.

E.x. $M(\mathbb{Q}_p\langle X \rangle) \supseteq \mathbb{Z}_p$

$M(\mathbb{C}_p\langle X, Y \rangle / (pY - X)) = p\,\mathcal{O}_{\mathbb{C}_p}$.

Thm (Weierstrass preparation) There is a bijection

$$\{\alpha \in \mathbb{C}_p : \|\alpha\|_p \leq 1\} \xleftrightarrow{1:1} \mathrm{Spec\,Max}(\mathbb{C}_p\langle X \rangle) = M(\mathbb{C}_p\langle X \rangle).$$

$$\alpha \longleftrightarrow \langle X - \alpha \rangle.$$

Def. For a k-alg. A, A is called a k-affinoid alg. if $\exists\, I \subseteq_{\text{ideal}} T_n(k)$, s.t.

$$A \simeq T_n(k)/I.$$

We can apply a norm on A by

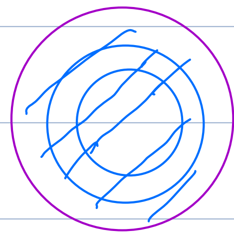
$$\|\bar{a}\| := \inf_{aI = \bar{a}} \|a\| \quad , \ \forall\, \bar{a} \in A \simeq T_n(k)/I.$$

Tate ring and k-Affinoid alg. have many nice properties, can be found at [Conrad].

Thm. ① T_n is noetherian, regular and UFD. $\forall\, m \in \mathrm{SpecMax}(T_n)$, the local ring $(T_n)_m$ has dim. n, and $[T_n/m : k] < \infty$.

② T_n is Jacobson: every prime ideal p of T_n is the intersection of maximal ideals containing it.

③ Any ideal $I \subseteq T_n$ is closed.



But there is a decomposition

disjoint!

$$\{\alpha \in \mathbb{C}_p : \|\alpha\| \le 1\} = \underset{\substack{\| \\ \mathrm{Spec\,Max}(\mathbb{C}_p\langle X, X^{-1}\rangle) \\ \| \\ \mathbb{C}_p\langle X_1, X_2\rangle / (X_1X_2 - 1)}}{\{\alpha \in \mathbb{C}_p : \|\alpha\| = 1\}} \cup \bigcup_{s>0} \underset{\substack{\| \\ \mathrm{Spec\,Max}(\mathbb{C}_p\langle \frac{X}{p^s}\rangle)}}{\{\alpha \in \mathbb{C}_p : \|\alpha\| \le p^{-s}\}}$$

All pieces are affinoid, this will give a terrible sheaf theory since we can get a global section of $\mathrm{SpecMax}(\mathbb{C}_p\langle X\rangle)$ which is non-zero at boundary but vanish inside. such function does not analytic from Weierstrass preparation theorem.

Tate avoid this by only consider "admissible open subset" and "admissible open cover" which is a Gro. topo. to some extent.

Def. For a k-affinoid alg. A, a rational domain U of $M(A)$ of the form

$$U\left(\frac{f_1, \cdots, f_n}{g}\right) = \{x \in M(A) : |f_i(x)| \le |g(x)| \ne 0\}$$

$f_i(x)$ = stalk of f_i at x, element in $k(x)$, has a unique norm from k.

$U \subseteq M(A)$ is admissible open if $U = \bigcup_{i \in I} U_i$, U_i are rational domain, and for any affinoid map $A \to B$, s.t. $\mathrm{im}\,(M(B) \to M(A)) \subseteq U$, then this image lies in finitely many U_i.

We can define admissible cover in a similar way, which makes

$$\{\alpha \in \mathbb{C}_p : \|\alpha\| \le 1\} = \{\alpha \in \mathbb{C}_p : \|\alpha\| = 1\} \cup \bigcup_{s>0} \{\alpha \in \mathbb{C}_p : \|\alpha\| \le p^{-s}\}$$

being not adm.

Def. $M(A)$ with adm. open subset and adm. open cover consist a site, which is called the G-topology on $M(A)$.

Thm. (Tate) $\mathcal{O}(U(\frac{f_1, \cdots, f_n}{g})) = A\langle T_1, \cdots, T_n\rangle / (gT_1 - f_1, \cdots, gT_n - f_n)$ defines a sheaf on this site.

Def. G-topologized space $(M(A), \mathcal{O})$ is called an affinoid space, a locally ringed G-topologized space is called rigid analytic space if it is locally an affinoid space.

Huber's adic space goes furthur:

Tate's rigid analytic space		Huber's adic space
CA		"topological" CA
Spec Max	($\xleftarrow{\text{supp}}$ - -)	Spa (valuation spectrum)
	rational domain (almost the same)	
affinoid alg.		(general) topological ring

affinoid alg. will always have a unit pseudo-uniformizer.

Huber ring

Def. A topological ring A is called a Huber ring if it contains an open subring A_0, with $I \underset{\text{ideal}}{\subseteq} A_0$, s.t.

① topology on A_0 (hence on A) is $\underbrace{I\text{-adic topology}}$.

② I is f.g.

$\{a+I^n\}_{\substack{a\in A\\ n\geq 0}}$ consist a top. basis

A_0 is called a ring of definition, I is called an ideal of definition.

$X \subseteq A$ is called bounded if for any open neighborhood U of 0, $\exists\ \underset{\text{open}}{0 \in V \subseteq A}$. s.t. $x \cdot v \in U$, $\forall x \in X$, $v \in V$.

Propo. A is a Huber ring, then $A_0 \subseteq A$ is a ring of def. $\Longleftrightarrow$ A_0 is open and bounded

Def. $f \in A$ is called power-bounded if $\{f^n\}_{n\geq 1}$ is bounded

$A^\circ := \{f \in A : f \text{ is power-bounded}\}$.

Lemma. Let A be a Huber ring with ring of def. A_0, $f \in A^\circ$, then $A_0[f]$ is a ring of def.

Only need to show $A_0[f]$ is open and bounded. since $I \subseteq A_0 \subseteq A_0[f]$, it is open. For any ideal of def. $I \subseteq A_0$, since $\{f^m\}$ is bounded, $\exists n>0$, s.t. $f^m \cdot I^n \subseteq I$, $\forall m \geq 1$. So that we have $A_0[f] \cdot I^n \subseteq I$, which means $A_0[f]$ is bounded.

Propo. Let A be a Huber ring, then

$$A^\circ = \bigcup_{\substack{A_0 \subseteq A \\ \text{ring of def.}}} A_0$$

A° is open and integrally closed in A.

$A_0 \subseteq A^\circ$, and $\forall f \in A^\circ$, ring of def. A_0, $A_0[f]$ is ring of def.

If $f \in A$ is integral over A°, that is

$$f^n + a_{n-1} f^{n-1} + \cdots + a_0 = 0, \quad a_i \in A^\circ.$$

Let $a_i \in A_0$, A_0 is a ring of def. Then $A_0' = A_0[a_{n-1}, \cdots a_0]$ is also a ring of def.

But then $A_0'[f]$ is open and bounded, hence $f \in A^\circ$.

Def. $f \in A$ is called topologically nilpotent, if for any $0 \in U \underset{\text{open}}{\subseteq} A$, $f^n \in U$ for $n \gg 0$.

$A^{\circ\circ} = \{ f \in A : f \text{ is top. nil.} \}$. Easy to see $A^{\circ\circ} \subseteq A^\circ$.

Propo. Let A be Huber ring, then

$$A^{\circ\circ} = \bigcup_{\substack{I \subseteq A \\ \text{ideal of def.}}} I.$$

and $A^{\circ\circ}$ is a radical ideal of A°.

E.x.	A	(A_0, I)	A°	$A^{\circ\circ}$
	$\mathbb{Q}_p$	$(\mathbb{Z}_p, p^2)$	$\mathbb{Z}_p$	(p)

top given by Gauss norm. $\leftarrow$ $\mathbb{C}_p[W]$ $\quad$ $(\mathcal{O}_{\mathbb{C}_p}[W],\ p\,\mathcal{O}_{\mathbb{C}_p}[W])$ $\quad$ $\mathcal{O}_{\mathbb{C}_p}[W]$ $\quad$ $\mathfrak{m}_{\mathbb{C}_p}[W]$

with t-adic top. $\leftarrow$ $k((t^{\frac{1}{p^\infty}}))$ $\quad$ $(k[[t^{\frac{1}{p^\infty}}]],\ t\,k[[t^{\frac{1}{p^\infty}}]])$ $\quad$ $k[[t^{\frac{1}{p^\infty}}]]$ $\quad$ $\mathfrak{m}_{k[[t^{\frac{1}{p^\infty}}]]}$.

with p-adic top. $\leftarrow$ $\mathbb{Q}$ $\quad$ $\mathbb{Z}_{(p)}$ $\quad$ $p\mathbb{Z}_{(p)}$

Next we will consider valuation spectrum of Huber rings. We need to consider higher rank valuation there instead of just norms.

Valuation, Cont(A).

We will use Γ denote an ordered abelian g.p., write the group operation mult'l.

Def. Valuation on a ring A is a map $|\cdot|: A \to \Gamma \cup \{0\}$, s.t.

$$|fg| = |f|\cdot|g|$$

$$|f+g| \le \max\{|f|, |g|\}$$

and $|0| = 0$. $|1| = 1$.

Note that Γ can embed into not just $\mathbb{R}^*_{>0}$, but also $(\mathbb{R}^*_{>0})^n$.

We will use $\|\cdot\|$ to denote the norm on A defining top., $|\cdot|$ denote some val on A.

inverse image of $|\cdot|$ is called the kernal of $|\cdot|$.

E.x. trivial val: $|a| = \begin{cases} 0, & a = 0 \\ 1, & a \neq 0 \end{cases}$, or $A \to A/\mathfrak{p} \xrightarrow{|\cdot|_{triv}} \{0,1\}$.

$|\cdot|_p$, $|\cdot|_\infty$ on $\mathbb{Q}$.

$\mathbb{Q}_p[[t]] \longrightarrow \mathbb{R}_{\ge 0}$, $\quad f(t) = \sum_{n \ge m} a_n t^n,\ a_m \neq 0 \longrightarrow r^{-m}$,

E.x. $A = \mathbb{C}_p\langle w\rangle$, $\quad |f|_\alpha := \|f(\alpha)\|_p$, $\forall \alpha \in \mathcal{O}_{\mathbb{C}_p}$.

has kernal $\langle w - \alpha\rangle$.

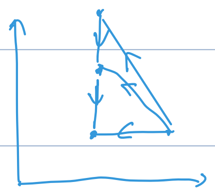
$\forall \alpha \in \mathcal{O}_{\mathbb{C}_p}$, let $f(w) = \sum_{n\geq 0} b_n (w-\alpha)^n$, then

$$|f|_{\alpha,r} := \max_{i\geq 0} \|b_i\| r^i, \quad 0<r<1.$$

In particular, $|\cdot|_{0,1}$ is the Gauss norm.

E.x. $\Gamma = \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, with 字典序, that is:

$$(a,b) > (c,d) \iff a>c, \text{ or } a=c \text{ and } b>d.$$



$\forall \alpha \in \mathcal{O}_{\mathbb{C}_p}$, $0<r\leq 1$, $0<\varepsilon<1$, $f(w) = \sum_{n\geq 0} b_n (w-\alpha)^n$

$$\mathbb{C}_p\langle w\rangle \longrightarrow \Gamma$$

$$f \longrightarrow |f|_{\alpha,r^-} := \max_{i\geq 0} (\|b_n\|_p \cdot r^n, \varepsilon^n)$$

$0<\varepsilon<1$, pick the lowest degree term.

$$f \longrightarrow |f|_{\alpha,r^+} := \max_{i\geq 0} (\|b_n\|_p \cdot r^n, \varepsilon^{-n})$$

pick the highest degree term.

Def. $|\cdot| : \underset{\text{top. ring}}{A} \longrightarrow \Gamma$ is a val. we call $|\cdot|$ is conti. if $\forall \gamma \in \Gamma$,

$$U_\gamma := \{a\in A \mid |a|<\gamma\} \underset{\text{open}}{\subseteq} A.$$

$\mathrm{Cont}(A) := \{\text{conti. val. on } A\}/\sim$, $\mathrm{Spv}(A) := \mathrm{Cont}(A_{disc})$.

E.x. $A \longrightarrow A/\mathfrak{p} \xrightarrow{|\cdot|_{triv}} \{0,1\}$ is conti. iff $\mathfrak{p} \underset{\text{open}}{\subseteq} A$.

Let $\mathbb{C}_p\langle w\rangle$ with top. defined by Gauss norm, $|\cdot|_\alpha$, $|\cdot|_{\alpha,r}$ is conti. for $0<r\leq 1$.

$|\cdot|_{triv}$ on $\mathbb{Q}_p$ is not conti.

$\forall a \in \mathbb{R}_{>0}$, note that for fixed $\alpha \in \mathcal{O}_{\mathbb{C}_p}$, $0 < r \leq 1$,

$$\|f\|_{\alpha,r} \leq a \implies \|f\|_{\alpha,r^-} \leq (a, \square)$$

$$\implies \quad U_a^{|\cdot|_{\alpha,r}} \subseteq U_a^{|\cdot|_{\alpha,r^-}}$$

Hence $|\cdot|_{\alpha,r^-}$ is also conti.

$\forall x \in \mathrm{Cont}(A)$, $f \in A$. $|f(x)| \in \Gamma_x$ is the valuation x at f.

Def. For $f_1, \cdots, f_n \in A$, $s \in A$, we can define

$$U\left(\frac{f_1, \cdots, f_n}{s}\right) := \{ x \in \mathrm{Spv}(A) \mid |f_i(x)| \leq |g(x)| \neq 0 \}$$

as open basis of $\mathrm{Spv}(A)$, and equip $\mathrm{Cont}(A)$ with induced top.

need $\langle f_1, \cdots, f_n, g \rangle$ is an open ideal

Lemma.

$$\mathrm{supp}: \mathrm{Spv}(A) \longrightarrow \mathrm{Spec}(A)$$
$$x \longmapsto \mathrm{supp}(x) = \{ f \in A : |f(x)| = 0 \}$$

is conti., and $\forall s \in A$, $\mathrm{supp}^{-1}(D(s)) = U\left(\frac{s}{s}\right)$ $\left(= \{ x \in \mathrm{Spv}(A) : |s(x)| \neq 0 \} \right)$.

$\forall \mathfrak{p} \in \mathrm{Spec}\, A$, there is a homeomorphism

$$\mathrm{supp}^{-1}(\mathfrak{p}) \simeq \mathrm{Spv}(\mathrm{Frac}(A/\mathfrak{p})).$$

$$\begin{tikzcd} A \arrow[r, "|\cdot|"] \arrow[d] & \Gamma \cup \{0\} \\ A/\mathfrak{p} \arrow[r, hook] \arrow[ur, dashed, "|\cdot|"] & \mathrm{Frac}(A/\mathfrak{p}) \arrow[u, dashed] \end{tikzcd}$$

extend $\implies$

E.x. trivial $\cdot$

p-adic $\cdot$ $\qquad$ $\cdot$ $\qquad$ $\mathrm{Spv}\, \mathbb{Z}_p$

$\cdot$ $\qquad$ $\cdot$ $\qquad$ $\mathrm{Spec}\, \mathbb{Z}_p$

0 $\qquad$ p

Def. $\gamma \in \Gamma \cup \{0\}$ is called cofinal if $\forall \delta \in \Gamma$, $\gamma^n < \delta$ for $n >> 0$.

Thm (continuity criterion) Let A be Huber ring, $|\cdot| : A \to \Gamma$ be a val., then TFAE

① $|\cdot|$ is conti.

② $f \in A$ is top. nilpotent $\Rightarrow$ $|f|$ is cofinal

③ Let (A_0, I) be a pair of def. $I = (f_1, \cdots, f_d)$, then $|f_i|$ are all cofinal and $|ff_i| < 1$. $\forall f \in A_0$

① $\Rightarrow$ ②: $\forall f \in A$ top. nil., $\gamma \in \Gamma$, then U_γ is open, hence $f^n \in U_\gamma$, $n >> 0$.

$\Rightarrow |f^n| < \gamma$, hence $|f|$ is cofinal.

② $\Rightarrow$ ③: since $I \subseteq A^{oo}$. $|f_i|$ are all cofinal. Since $\forall f \in A_0$, $ff_i \in I$,

we have $|ff_i| < 1$.

③ $\Rightarrow$ ①: $\forall \gamma \in \Gamma$, let n large enough s.t. $|f_i|^n < \gamma$, $1 \leq i \leq d$. We will

prove $I^{nd+1} \subseteq U_\gamma$ to show that $|\cdot|$ is conti.

I^{nd+1} generated by $g = f f_1^{m_1} \cdots f_d^{m_d}$, $f \in A_0$, $m_1 + \cdots + m_d = nd+1$.

easy to see $g = f' \cdot f_i \cdot f_j^n$, and $|g| < |f_j^n| < \gamma$.

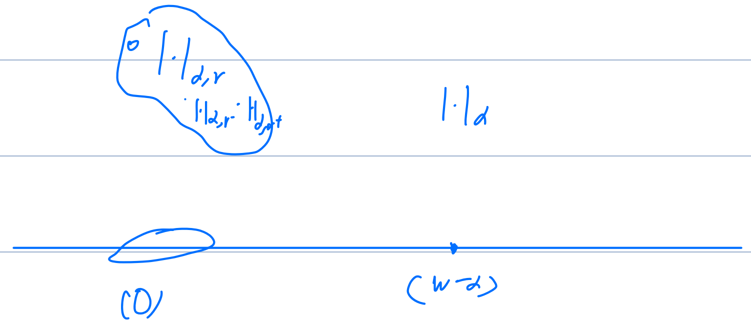
E.x. $A = \mathbb{C}_p\langle w \rangle$, we can choose $(\mathcal{O}_{\mathbb{C}_p}\langle w \rangle, p\mathcal{O}_{\mathbb{C}_p}\langle w \rangle)$ as ring of def,

from above, val. $|\cdot|$ is conti. $\Leftrightarrow$ $|p|$ is cofinal and $|p\mathcal{O}_{\mathbb{C}_p}\langle w \rangle| < 1$.

$\Rightarrow |\cdot|_{\alpha, r^+}$ is conti.

On the other hand, $|\cdot|_{\alpha, r^+}$ and $|\cdot|_{\alpha, r^-}$ are in the $\overline{\{|\cdot|_{\alpha,r}\}}$ under the top. on $\mathrm{Cont}(A)$

given above.



$$\begin{array}{c} \mathrm{Cont}\,\mathbb{C}_p(w) \\ \downarrow \\ \mathrm{Spec}\,\mathbb{C}_p[w] \end{array}$$

Huber's unit closed disc model is

$$\mathcal{O}_{\mathbb{C}_p} \overset{\alpha \longrightarrow |\cdot|_\alpha}{\subseteq} D := \{ x \in \mathrm{Cont}(\mathbb{C}_p\langle w\rangle) \mid |w(x)| \leq 1 \}.$$

$$\underset{\neq}{} \quad \mathrm{Cont}(\mathbb{C}_p\langle w\rangle), \text{ since } |\cdot|_{\alpha, r^+} \notin D.$$

which will avoid the question in the beginning: let $V_1 = \{ x \in D : |w(x)| = 1 \}$,

$$V_{<1} = \{ x \in D : |w(x)| < 1 \} \supsetneqq \bigcup_{n>0} U\left(\frac{w^n}{p}\right) \rightsquigarrow \{ x \in D : |w(x)|^n < |p(x)|,\ \forall n > 0 \}$$

$$\overset{\cup}{|\cdot|_{0,1^-}},$$

but $|w|^n_{0,1^-} = (1, \varepsilon^k)^n > |p|_{0,1^-} = (\tfrac{1}{p}, 1)$

$$\Longrightarrow \quad D \neq V_1 \cup \bigcup_{n>0} U\left(\frac{w^n}{p}\right)$$

Huber pair.

Def. Let A be a Huber ring, $A^+ \subset A^\circ$ is open and integral closed in A, then (A, A^+) is called a Huber pair, and

$$\mathrm{Spa}(A, A^+) := \{ x \in \mathrm{Cont}(A) : |f(x)| < 1,\ \forall f \in A^+ \}$$

is the adic spectrum corr. to it.

We can also equip top. on $\mathrm{Spa}(A, A^+)$ from $\mathrm{Cont}(A)$, with open basis

$$\underbrace{R\left(\frac{f_1, \cdots, f_n}{g}\right)}_{\text{rational open subset.}} := \{ x \in \mathrm{Spa}(A, A^+) \mid |f_i(x)| \leq |g(x)| \neq 0,\ 1 \leq i \leq n \},$$

$\langle f_1, \cdots, f_n, g\rangle$ is open ideal

Thm. (Huber) For any Huber pair (A, A^+), $\mathrm{Spa}(A, A^+)$ is a spectral $\overset{(\Leftrightarrow\, \mathrm{qcqs})}{\text{space}}$.

and all rational open subset are q.c.

proof can refer to Morel, Adic space. Corollary III.2.4

tensor product :

$$\begin{tikzcd} (B, B^+) \arrow[dr] & & (C, C^+) \arrow[dl] \\ & (A, A^+) & \end{tikzcd}$$

$$\rightsquigarrow (B \otimes C,\ \mathrm{im}(B^+ \otimes_{A^+} C^+ \to B \otimes C)).$$

complement: Given ring of def (A_0, I), $\hat{A} = \varprojlim A/I^n$,

$(A, A^+) \rightsquigarrow (\hat{A}, \hat{A}^+)$, and we have

Thm. (Huber) $\mathrm{Spa}(A, A^+) \xrightarrow{\simeq} \mathrm{Spa}(\hat{A}, \hat{A}^+)$. [HLV24, Chapter 1, 4.3, 4.4]

Thm (Cadic - Null___) Given Huber pair (A, A^+),

$$A^+ = \{ f \in A : |f(x)| \le 1,\ \forall x \in \mathrm{Spa}(A, A^+) \}.$$

[HLV24, Chapter 1, Thm 4.2].

Ex. Let $A = \mathbb{C}_p \langle w \rangle$,

$$D = \{ x \in \mathrm{Cont}(A) : |w(x)| \le 1 \}$$

adic unit disc over $\mathbb{C}_p$. $= \mathrm{Spa}(\mathbb{C}_p\langle w \rangle, \mathcal{O}_{\mathbb{C}_p}\langle w \rangle)$

easy to see $|\cdot|_{\alpha, r^+}$ not in $\mathrm{Spa}(\mathbb{C}_p\langle w \rangle, \mathcal{O}_{\mathbb{C}_p}\langle w \rangle)$

Ex. Let $A = \varprojlim_n A/I^n$, I f.g., then the trivial val. in $\mathrm{Spa}(X, X)$ is the same as $\mathrm{Spf}(A)$.

Structure (pre)-sheaf.

Now given $X = \mathrm{Spa}(A, A^+)$, we want to give a sheaf of complete topological ring $\mathcal{O}_X \supseteq \mathcal{O}_X^+$ on X s.t.

$$\mathcal{O}_X(X) = \hat{A}, \quad \mathcal{O}_X^+(X) = \hat{A}^+.$$

For any rational subset $U = R\left(\frac{f_1, \cdots, f_n}{g}\right)$, define (A_U, A_U^+) to be the completion of Huber pair

$$\left(\underbrace{A[\tfrac{1}{g}]}, \ A^+[\tfrac{f_1}{g}, \cdots, \tfrac{f_n}{g}]^{\sim} \right)$$

integral closure.

use $(A_0[\frac{f_1}{g}, \cdots, \frac{f_n}{g}], \ I A_0[\frac{f_1}{g}, \cdots, \frac{f_n}{g}])$ be a pair of def.

Def. Given $X = \mathrm{Spa}(A, A^+)$, the sheaf on rational subsets

$$\mathcal{O}_X(U) = A_U, \quad \mathcal{O}_X^+(U) = A_U^+$$

is the structure presheaf of X. If $\mathcal{O}_X$ on X is a sheaf then (A, A^+) is called a sheafy Huber pair.

Lemma. (A, A^+) is sheafy then $\mathcal{O}_X^+$ is a sheaf.

From adic Null____,

$$\mathcal{O}_X^+(U) = \{ a \in \mathcal{O}_X(U) : |a(x)| \le 1, \ \forall x \in A_U^+ \}.$$

Thm. Let (A, A^+) be Huber pair s.t. one of following

1. A is discrete.
2. A has a noetherian ring of def. formal scheme.
3. A is a strongly noetherian Tate ring
   - strongly noetherian: $A\langle T_1, \cdots, T_n \rangle$ is also noetherian for any n.
   - Tate ring: has a top. nilpotent unit.
4. (A, A^+) is stably uniform
   - stably: $\forall U \subseteq_{\text{rational}} \mathrm{Spa}(A, A^+)$, $\mathcal{O}_X(U)$ is uniform.
   - uniform: A is a Tate ring, and A° is bounded.
   - perfectoid case

proof can be found at Morel, Adic space, Chapter IV.

Point of $\mathcal{D}$.

We will use x_α to denote $|\cdot|_\alpha$ in $\mathcal{D}$.
$x_{\alpha,r}$ $|\cdot|_{\alpha,r}$, etc.

Let $f(w) = \sum_{n\geq 0} a_n w^n = \sum_{n\geq 0} b_n(w-\alpha)^n \in \mathbb{C}_p\langle w\rangle$, $\alpha \in \mathcal{O}_{\mathbb{C}_p}$.

recall

$$|f(x_\alpha)| = \|f(\alpha)\|_p \quad \text{with kernal } \langle w-\alpha\rangle.$$

$$|f(x_{\alpha,r})| = \max \|b_n\| r^n ,$$

Let $D_r(\alpha) = \{ \alpha' \in \mathbb{C}_p : |\alpha-\alpha'| \leq r\}$. then we have

$$|f(x_{\alpha,r})| = \sup_{\alpha' \in D_r(\alpha)} \|f(\alpha')\|_p$$

Let $\{D_i\}$ be an decreasing sequence of disc in $\mathcal{O}_{\mathbb{C}_p}$, then

$$f(x_{D_\cdot}) := \inf_{i\geq 0} |f(x_{D_\cdot})|$$

is also an element in $\mathcal{D}$, in particular, $\bigcap_{i\geq 0} D_i$ can be empty.

Type		supp(x)	val. g.p.	A_x/m_x	
1	x_α	$\langle w-\alpha\rangle$	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}_p}$	closed
2	$x_{\alpha,r}(r\in p^{\mathbb{Q}})$	(0)	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}_p}(t)$	non-closed
3	$x_{\alpha,r}(r\notin p^{\mathbb{Q}})$	(0)	$p^{\mathbb{Q}} r^{\mathbb{Z}}$	$\overline{\mathbb{F}_p}$	c
4	$x_{D_\cdot}$		$p^{\mathbb{Q}}$	$\overline{\mathbb{F}_p}$	c
5	$x^\lambda_{\alpha,r}(r\in p^{\mathbb{Q}})$		$p^{\mathbb{Q}} \times (\frac{1}{2})^{\mathbb{Z}}$	$\overline{\mathbb{F}_p}$	c

$\lambda \in \mathbb{A}^1(\overline{\mathbb{F}_p})$. $|f(x^\lambda_{\alpha,r})| := (|f(x_{\alpha,r})|, (\frac{1}{2})^{\mathrm{ord}_\lambda \bar{f}})$, $f \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle \mapsto \bar{f} \in \overline{\mathbb{F}_p}\langle w\rangle$.

$$|p(x_{\alpha,r}^{\lambda})| := (\frac{1}{p}\ ,1)$$

Easy to see that $x_{\alpha,r}^{\circ} = |\cdot|_{\alpha,r^-}$ from above.