- Prologue : from congruences between modular forms to eigenvarieties

21.07.2025

- Local geometry of the eigencurve

23.07.2025

§1. Congruences between modular forms
3 examples regarding Ramanujan Δ

Ex 1. (Ramanujan, 1916)

$$\tau(p) \equiv 1 + p^{11} \ (691), \quad \forall p.$$

Indeed. $M_{12}(1) = \langle \Delta, G_{12} \rangle \Rightarrow G_6^2$ must be a linear combination of the two. $\rightsquigarrow$

$$\frac{691}{65520} \cdot 504^2 G_6^2 = G_{12} - \frac{756}{65} \Delta$$

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} (mz+n)^{-k} = \underbrace{-\frac{B_k}{2k}}_{-\frac{1}{2}\zeta(1-k)} + \sum_{n\geq 1} \sigma_{k-1}(n) q^n, \quad q = e^{2\pi i z}$$

$$\sigma_{k-1}(n) = \sum_{d\mid n} d^{k-1}$$

$$\Delta(q) = q \prod_{n\geq 1} (1-q^n)^{24}$$

$$\Rightarrow G_{12} \equiv \Delta \ (691).$$

Ex 2. (Wilton, 1930)

$$q \prod_{n\geq 1}(1-q^n)^{24} \underset{(23)}{\equiv} \overset{=\eta}{q^{1/24} \prod_{n\geq 1}(1-q^n)} \cdot q^{23/24} \prod_{n\geq 1}(1-q^{23n})$$

$$\underset{(23)}{\equiv} \underbrace{\frac{1}{2} \sum_{u,v\in\mathbb{Z}} \left(q^{u^2+uv+6v^2} - q^{2u^2+uv+3v^2}\right)}$$

Hecke eigenform wt 1.

Euler identity:

$$\prod_{n\geq 1}(1-q^n) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3n^2+n}{2}}$$

$$\tau(p) \equiv ? \ (23)$$

Assoc. Galois rep. : $\mathbb{Q}(\sqrt{-23})$ has cls number 3. its Hilbert cls field H is obtained by adjoining a root of

$$f(x) = x^3 - x - 1$$

which has discriminant -23.

$$G_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(H|\mathbb{Q}) \simeq S_3 \overset{\text{unique 2-dim. irrep}}{\longrightarrow} GL_2(\mathbb{C})$$

$\rightsquigarrow$ cong. cls of $\tau(p)$ mod 23 $\leftrightarrow$ splitting behavior of p in $H|\mathbb{Q}$.

<u>Ex</u> 3.

$$q \prod_{n\geq 1}(1-q^n)^{24} \equiv \underbrace{q\prod_{n\geq 1}(1-q^n)^2(1-q^{11n})^2}_{\text{wt 2 cusp form}} \quad (11)$$

$$\leftrightarrow E: \ y^2 + y = x^3 - x^2 - 10x - 20$$

$\Rightarrow$

$$\tau(p) \equiv p+1 - |E(\mathbb{F}_p)| \quad (11)$$

cannot be made explicit as Ex 1 & 2

$\bar{\rho}_E : G_{\mathbb{Q}} \twoheadrightarrow GL_2(\mathbb{F}_{11})$ not solvable.

These examples are of different flavours, & illustrate different but related phenomena that arise in the p-adic theory of modular forms.

1. cusp form & Eisenstein series of the same wt $\rightsquigarrow$ Iwasawa theory & p-adic L-fcns

2. & 3. cusp forms of different wts

wt 1. $\rightsquigarrow$ Deligne - Serre

ell. curves

§2. p-adic modular forms à la Serre

Restricting to level 1 here, but can be lifted in the framework of Katz.

For any formal power series

$$f(q) = a_0 + a_1 q + a_2 q^2 + \cdots \in \mathbb{Q}_p[[q]],$$

define $v_p(f) = \inf_n (v_p(a_n))$. A <u>$p$-adic modular form</u> $f \in \mathbb{Q}_p[[q]]$ is s.t. $\exists$ a seq. $f_i \in S_{k_i}(SL_2(\mathbb{Z}))$ w/ rat'l Fourier coeffs satisfying

$$v_p(f(q) - f_i(q)) \to \infty, \text{ as } i \to \infty.$$

In other words, the space of p-adic modular forms is the p-adic closure of the set of modular forms.

<u>Prop</u>: f, g two modular forms of wt k, ℓ, level 1, non-zero & normalized s.t. $v_p(f) = 0$. Suppose that

$$v_p(f-g) \geq m$$

for some $m \in \mathbb{Z}_{>0}$. Then

$$k \equiv \ell \ \ ((p-1)p^{m-1}), \quad \text{if } p \geq 3$$

$$k \equiv \ell \ \ (2^{m-2}), \quad p = 2$$

$\Rightarrow$ every p-adic modular form f has a well-def. wt

$$k := \varprojlim_i k_i \in \underbrace{\mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}}_{\text{is } \mathbb{Z}_p^\times}$$

<u>Thm</u>: Suppose we have a seq. of p-adic mod. forms of wt k_i,

$$f_i(q) = a_0^{(i)} + a_1^{(i)} q + \cdots$$

s.t. 1) $a_n^{(i)} \to a_n \in \mathbb{Q}_p$ uniformly for $n > 0$ &
2) $k_i \to k \neq 0$

$\Rightarrow a_0^{(i)} \to a_0$ &

$$f(q) = a_0 + a_1 q + a_2 q^2 + \cdots \in \mathcal{O}_p[[q]]$$

is a p-adic mod. form.

Serre used this idea of "inheriting" congruences for the constant terms of Eisenstein series from the much more elementary congruences between their higher coeffs. to give another construction of p-adic L-fcns :

$$G_k^* = (1-p^{k-1}) G_k$$

$$= (1-p^{k-1})\zeta(1-k) + \sum_{n\geq 1} \sigma_{k-1}^*(n) q^n, \quad \sigma_{k-1}^*(n) = \sum_{\substack{d\mid n \\ p\nmid d}} d^{k-1}$$

"p-stabilization" $\quad$ wt k, level $\Gamma_0(p)$

congruence relations :

$$\underbrace{\qquad\qquad}_{?} \qquad \underbrace{\qquad\qquad}_{} \quad \begin{array}{l} k \equiv k' \ ((p-1)p^n) \\ d^{k-1} \equiv d^{k'-1} \ (p^{n+1}) \end{array}$$

can p-adically interpolate

$d \longmapsto d^{k-1}$, k varying p-adically

YES

- Kummer congruences
- Kubota-Leopoldt p-adic L-fcn ; Iwasawa ~ses
- Serre Thm above.

($p \mid d$ ⇝ trouble)

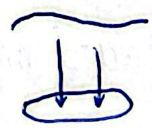
Γ, $x \longmapsto x^k := e^{k \log x}$

↑ does not behave well on all of $\mathbb{Z}_p$

· "measure" on $\mathbb{Z}_p^\times$

⇝ $E(z) = \sum_{n\geq 0} A_n q^n \in \mathcal{Q}(\mathbb{Z}_p^\times)[[q]]$ w/ "Λ-adic mod form"

- A_0 "pseudo-measure"
- $A_n \in \Lambda(\mathbb{Z}_p^\times)$



Hida : can do this for <u>ordinary</u> modular forms + much more.

<u>wt space</u> : $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) = (p-1)$ open unit balls in $\mathbb{C}_p$

$\mathbb{Z} \hookrightarrow \mathcal{W}(\mathbb{C}_p)$, $x \mapsto x^k$

↑ functor on rigid analytic spaces rep^{le} by a rigid space.

measure on $\mathbb{Z}_p^\times$ ⟷ bdd rigid analytic fcn on $\mathcal{W}$

⇝ $E \in \mathcal{O}(\mathcal{W})[q^{-1}][[q]]$ ↑, p-adic interpolation over the wt space

§3. Katz's formulation.

Recall: $M_k(\Gamma) = H^0(X_\Gamma, \omega^k)$ ($X_\Gamma = \Gamma\backslash\mathcal{H} \cup \{\text{cusps}\}$)

⌜ can make this in an alg. geo. way

- X_Γ = "moduli space of ell. curves + level strs" / $\mathbb{Q}$
- $\mathcal{E} \xrightarrow{\pi} X_\Gamma$ univ. ell. curve
- $\omega = \pi_* \Omega^1_{\mathcal{E}/X}$ "automorphic line bundle"

} comes from rep^{le} functors $\mathrm{Ell}/R_0 \to \mathrm{Set}$, $E/S \mapsto$ level strs $\rightsquigarrow \mathcal{E}/X$ Katz & Mazur

fiberwise: sheaf of inv. diff. on ell. curves

$\Gamma = \Gamma_1(N)$, $X_1(N) / \mathbb{Z}[\frac{1}{N}]$ proj. curve. explicit def eqn.? [DS05] ch. 7

$S = \mathrm{Spec}\ \mathbb{Z}[j, j^{-1}(j-1728)^{-1}]$

e.g. $Y^2Z + XYZ = X^3 - 36(j-1728)^{-1}XZ^2 - (j-1728)^{-1}Z^3$; Tate curve

Upshot: can work w/ p-adic or mod p coeffs., not just $\mathbb{C}$. ⌟

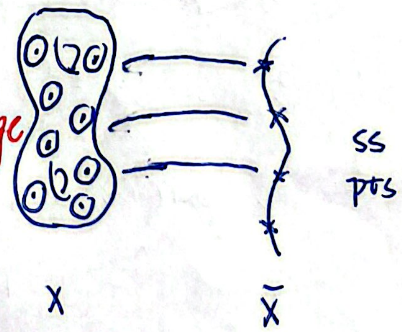
<u>Fact</u>: There is a special mod p mod. form $Ha \in H^0(\bar{X}(1)/\mathbb{F}_p, \omega^{p-1})$ w/ q-expansion const. 1 which has simple zeros precisely at ss ell curves.

Observe: $E_{p-1} \equiv Ha \equiv 1\ (p)$, so the Eisenstein series of wt $p-1$ is a lift of Ha. Now, in the world of p-adic mod forms, $E_{p-1}^{p^n} \to 1$ & so $E_{p-1}^{p^n-1} \to E_{p-1}^{-1}$ $\rightsquigarrow$ we are making E_{p-1} inv.

<u>Katz</u>: $M_k(N)$ "=" $H^0(X_1(N)^{ord}, \omega^k)$ ← not legal in the language of A.G. (schemes)

- rule-based defn.
- passing to rigid analytic spaces

$M_k(N) = H^0(X_1(N)^{rig, ord}, \omega^k)$.



§4. Overconvergent forms & the eigencurve.

Problem:

- Too many p-adic mod. forms to have a good spectral theory for the U_p-operator

  e.g. $\forall \alpha \in \overline{\mathbb{Q}}_p$ w/ $v(\alpha) > 0$, $\exists f$ s.t. $U_p f = \alpha f$

  (start w/ f ord., consider

  $$f' := f + \alpha V_p f + \alpha^2 V_p^2 f + \cdots$$

  $\ulcorner V_p(\sum a_n q^n) = \sum a_n q^{np}$; $U_p(\sum a_n q^n) = \sum a_{np} q^n$

  so $U_p \cdot V_p = \mathrm{id}$ $\lrcorner$

Soln: Overconv. mod forms $M_k^{\dagger, \le v}(N) := H^0(X^{\le v}, \omega^k) \longleftrightarrow M_k(\Gamma_1(N))$

[Coleman, 90s]

$\circlearrowleft$ U_p compact. "slope" Mention: Coleman's classicality

$\Downarrow$ eigenvariety machine [Buzzard, '06]

$\mathcal{E}$ eigencurve = moduli of systems of Hecke eigenvalues of overconv. forms

$w \downarrow$

$\mathcal{W}$ wt space.

1) $w : \mathcal{E} \to \mathcal{W}$ loc. quasi-fin. & flat $\Rightarrow$ $\mathcal{E}$ is equidim 1.

2) $\forall\, \mathcal{C} \subset \mathcal{E}$ irr. component $w(\mathcal{C})$ = complement of fin. many pts

3) cls pts are dense in $\mathcal{E}$

4) $\mathcal{E}$ is red.

5) $\mathcal{E}$ carries a family of (pseudo)-Galois reps: $\exists$ cts fcn

$$T : G_{\mathbb{Q}} \to \mathcal{O}^+(\mathcal{E})$$

s.t. $\forall x \in Z$, T_x is the trace of the Galois rep assoc. to x.

Open questions :

- Finiteness of conn components ; near the boundary of the wt space (Liu - Wan - Xiao '17 ;
- Properness of the wt map (Diao - Liu '16)
- Ghost conjecture (Liu - Truong - Xiao - Zhao '22). LWX '17 ;

... [BG16]

Today : $\mathcal{E} \xrightarrow{w} \mathcal{W}$

x cls. ($\leadsto$ new form of wt k, level N)

1) is $\mathcal{E}$ sm. at this pt ?

2) is w étale at this pt ?

- wt $k \geq 2$,
  - p-reg. & non-critical (Hida, Coleman)

    $\Updownarrow$ $\mathcal{L}_p(w^*) \neq 0$

    no forms are known to be p-irreg. (conj. x exists)

    sm. $\vee$ étale sub. std conj. in Gal. coh

  - evil Eisenstein series [BC06] — cusp form as a p-adic Φ mod. form.

    sm. $\vee$ $\times$ étale, ram. index $=2$

  - CM pts [Bel11] sub. conj. — e.g. $E_k(z) - E_k(pz)$ level $\Gamma_0(p)$ vanishing at cusp ∞, but not 0.

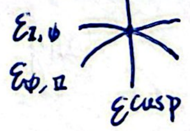
    $\rho_f \simeq \rho_f \otimes \chi_{K/\mathbb{Q}}$ $\simeq \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \chi$; E has CM $\Leftrightarrow$ ρ_E has CM in the above sense.

  - non-CM pts [BE10]

    ram. $\Leftrightarrow$ $\exists$ ord. companion form

    conj. $\times$ exist such pts

- wt $k=1$



Ferrero - Greenberg

  - p-reg. [BD16] sm. + étale
  - p-irreg. Eisenstein series [BDP22] — $\mathcal{E}^{cusp}$ intersects transver-sally each of the two Eisenstein components $\Downarrow$ p-adic L-fcns simple zero at $s=0$

    $f(z) = E_1(\mathbb{1}, \psi)(z) \in \mathcal{E}_{1,\phi}, \mathcal{E}_{\phi,1}$

    $\psi(p) = 1$

  - p-irreg. CM pts [BD21]
  - Remaining cases [BMP25] Complete understanding

§5.

Strategy of [BD16]:

1. Introduce a deformation problem $\mathcal{D}$ for ρ rep. by R s.t.

$$R \twoheadrightarrow T \quad \text{complete loc. ring of } \mathcal{E} \text{ at } x.$$

2. Computation of the tangent space $t_{\mathcal{D}}$ to $\mathcal{D}$ $\longleftrightarrow$ tangent space of $\mathcal{E}$ at x

$\Rightarrow \dim t_{\mathcal{D}} = 1.$ ☆

$\Rightarrow R \simeq T \Rightarrow$ sm.

3. Etaleness : study the alg. of the fiber of the wt map at x

$\rightsquigarrow$ another deformation problem. $\mathcal{D}'$

$t_{\mathcal{D}'} \longleftrightarrow$ tangent space at that pt to the fiber of the eigencurve over the wt space.

⌜ Rough picture :

Galois side		Auto. side
Galois rep.	$\longleftrightarrow$	mod. form
family of Gal rep	$\longleftrightarrow$	eigencurve
deformation.	$\longleftrightarrow$	near a pt
tang sp.	$\longleftrightarrow$	tang sp.
fixing det.		fiber over a pt (fixing the wt) or central char.

easier to understand.

higher dim?
Belliache
Dimitrov
Betina
...

⌟

- Galois coh.
- $H^1(H, \overline{\mathbb{Q}}_p) \overset{\text{CFT}}{=} \mathrm{Hom}(G_H, \overline{\mathbb{Q}}_p) \leftarrow$ str of $\overline{\mathbb{Q}}_p[G]$-module. Baker - Brumer

§6. Construction of the eigencurve.

∘ 6.1 Eigenvariety machine

input data — wt space; spaces of p-adic auto forms (e.g. Coleman's overconv. forms)

$\downarrow$

$\mathcal{Z} \to \mathcal{W}$ spectral var — parametrizes non-zero eigenvalues of one special Hecke operator (e.g. U_p)

$\downarrow$

$\mathcal{E} \to \mathcal{Z} \to \mathcal{W}$ eigenvar. — fin. / $\mathcal{Z}$, parametrizes systems of eigenvalues

Toy example:

- R red. affinoid K-alg.
- M f.g. proj. R-mod.
- $\mathbb{T}$ comm. R-alg. w/ an R-alg. hom. $\mathbb{T} \to \mathrm{End}_R(M)$
- $\phi \in \mathbb{T}$, id. w/ end. $\phi : M \to M$. Assume ϕ inv.

  $P(T) = \det(1 - T\phi) = 1 + \cdots \in R[T]$

- Let $\mathcal{Z}$ denote the zero locus of $P(T)$, regarded as a fcn. on $\mathrm{Max}(R) \times \mathbb{A}^1$.

  Then $R[T]/(P(T))$ is a fin. R-alg. $\Rightarrow$ affinoid alg. $\Rightarrow$ $\mathcal{Z} = \mathrm{Max}(-)$.

  is the rigid analytic space assoc. to $R[T]/(P(T))$

- $\mathbb{T}(\mathcal{Z}) :=$ image of $\mathbb{T}$ in $\mathrm{End}_R(M)$ aff. $\rightsquigarrow$ $\mathrm{Max}(\mathbb{T}(\mathcal{Z})) =: \mathcal{E}$

  Now, by Hamilton-Cayley $\phi^{-1} \in \mathbb{T}(\mathcal{Z})$, & $\exists$ nat'l map

$$\begin{aligned} R[T]/(P(T)) &\to \mathbb{T}(\mathcal{Z}) \\ T &\mapsto \phi^{-1} \end{aligned}$$

  So $R \to R[T]/(P(T)) \to \mathbb{T}(\mathcal{Z})$ gives $\underset{\text{eigenvar.}}{\mathcal{E}} \to \underset{\text{spectral var.}}{\mathcal{Z}} \to \mathcal{W}$

"e.g." $R = \mathbb{C}$, $M_R = M_k(N)$, $\mathbb{T}_R$ = Hecke alg. (tensor w/ $\mathbb{C}$) $\circlearrowleft$ faithfully

$$\mathrm{Max}(\mathbb{T}_R) \overset{1-1}{\longleftrightarrow} \text{hom. } \lambda : \mathbb{T}_R \to \mathbb{C} \overset{1-1}{\longleftrightarrow} \text{Hecke eigensystems}$$

* M not f.g.
* φ not inv. (cpt)
} need non-Arch. fun'e analysis

Goal:

- Learn the language of adic spaces
- Work through the details of the eigenvariety machine
- Translate into the language of adic spaces

0.2 Further developments

What about general red gps G, not just GL_2?

- The eigenvariety machine still works. So only need to feed in approp. inputs.

① Coleman's geometric approach.
using overconv. forms
(does not work well in general?)

② G cpt mod center.

" overconv. mod. symbols "

$\updownarrow$ cousin.

gp-coh. avatar of overconv. p-adic. mod forms

③ Completed coh. (Emerton). more rep-theoretic flavour.