LSGNT*: Iwasawa Theory

Haoran Liang, Wenhan Zhang

August 9, 2025

Goals. The purpose of this semi-formal study group is twofold: Firstly, we plan to work through Coates & Sujatha's classic "Cyclotomic Fields and Zeta Values" [CS06] in detail, so as to digest the proof of the Iwasawa Main Conjecture for $\mathbb{Q}$ via Euler systems[1]. Following this, we would then like to expose ourselves to more advanced materials such as

- Iwasawa theory for elliptic curves/Galois representations
- p-adic L-functions and eigenvarieties
- special value conjectures
- (φ, Γ)-modules
- $\cdots$

depending on interest (but yeah, we are interested in almost everything). Attractive looking materials include:

- Washinton's standard textbook on classical Iwasawa theory [Was21], as well as various fantastic course notes, e.g. Sharifi's notes and Williams I (with Jacinto) and II - as complementary texts to [CS06] and much more;
- Skinner's CMI summer school 2009 notes[2] [Ski09], maybe together with Bellaïche's notes [Bel09] on Bloch–Kato?
- Ochiai's recent three volumes [Och23], [Och24] (third one not yet published) - provide a compass for the world of Iwasawa theory that includes the new trends of Iwasawa theory for p-adic Galois representations;
- Colmez's Tsinghua notes [Col04] - constructions of p-adic L-functions using the theory of (φ, Γ)-modules of Fontaine;
- Part III of Bellaïche's book [Bel21] - construction and study of p-adic L-functions in family via the eigencurve;
- Tate's classic on Stark's conjecture [Tat84];
- Important papers by Iwasawa himself, Ribet, Greenberg, Kato, just to name a few

*Stands for "London Study Group on Number Theory".

[1] This proof, owe to Karl Rubin, is different from the original one by Mazur–Wiles (1984).

[2] Not sure if this is still available online..., but don't worry, we have managed to get a copy from MSE.

- $\cdots$

As a corollary, even summer break isn't going to stop us from carrying on with our study group next term, is it?

Time & Location. 14:00-16:00 on Tuesdays, (presumably) S3.40 at King's Strand.

Introduction. The organizers are too lazy to plagiarize from other sources. See [Kat07].

Tentative schedule. The following is an outline and suggestion for each talk. The organizers are happy to discuss the materials and organization of the talks, especially if you find too many/few topics are assigned in one talk, etc.; we are flexible with the schedule.

Talk 0. (Prof. David Burns) Overview of Iwasawa theory.

Main goal I: classical Iwasawa theory

Talk 1. (Haoran) Following chapter 1 of [CS06], introduce the main players in classical Iwasawa theory and fix notations - in particular, the highlight of the first few weeks will be to make sense of Theorem 1.4.2 and its proof (Proposition 4.24); motivate and summarize chapters 2-3 [CS06], the study of local units (especially Theorem 2.1.2, known as Coleman power series) and the Kubota–Leopoldt p-adic L-function (as a pseudo-measure on the Iwasawa algebra $\Lambda(\mathbb{Z}_p)$).

Talk 2. (Wenhan) Discuss Coleman's proof of Theorem 2.1.2 - along the way, introduce the trace and norm operators on the power series ring $R = \mathbb{Z}_p[[T]]$; study properties of logarithm derivative and establish the exact sequence (Theorem 2.5.2), which will be very important to us. Bonus: touch on section 2.6 if time permits.

Talk 3. (Wenhan) Following chapter 3 of [CS06] and sections 1-3 of Williams I, introduce p-adic measures and Iwasawa algebras and explain why they are the same; identify the Iwasawa algebra $\Lambda(\mathbb{Z}_p)$ with the power series ring $R = \mathbb{Z}_p[[T]]$ via the Mahler transform and discuss restriction of measures - please do amuse the audience with some explicit cute calculations; prove the Fundamental Exact Sequence (Theorem 3.5.1).

Note for speakers of talks 2-3: Presenting a full proof of Theorem 2.5.2 in talk 2 might be somewhat too ambitious, and if this is done Theorem 3.5.1 then becomes an immediate corollary, thus making the last part of talk 3 a bit too dry. Feel free to reorganize the materials accordingly.

Talk 4. (Simon) Following sections 4.1-4.4 of [CS06], introduce p-adic zeta functions and prove Iwasawa's Main Theorem (Theorem 4.4.1); discuss its relation with the Main Conjecture (section 4.5). Bonus: discuss the general case of p-adic interpolating Dirichlet L-functions and Eisenstein series (sections 4-5 of Williams I).

Talk 5. (Teymour) Finish the rest of chapter 4 of [CS06]. Cute bonus: discuss the following algebraic incarnation of Class Number Formula[1]: Let $F_n = \mathbb{Q}(\zeta_{p^n})^+$ be the maximal totally real subfield of $\mathbb{Q}(\zeta_{p^n})$, and $V_n = \mathcal{O}_{F_n}$ its ring of algebraic integers. Consider the subgroup D_n generated by cyclotomic units (see Definition 4.3.1, [CS06]), we have

$$[V_n : D_n] = \#\mathrm{Cl}(F_n).$$

Note for speakers of talks 4-5: Again, feel free to reorganize the materials accordingly.

[1] Are there good references for this theorem?

Talk 6 (Dr. Dominik Bullach) Overview of Euler systems.

Talk 7 (Wenhan) Following chapter 5 of [CS06], introduce Euler systems and discuss their basic properties. In particular, check that the map (5.1) indeed gives an Euler system. Define the Kolyvagin derivative (5.7) as well as the Kolyvagin class (Proposition 5.4.5). Sketch a proof of the Factorization Theorem (Theorem 5.4.9).

Talks 8-9 (Haoran, Wenhan) Finish the proof the Iwasawa Main Conjecture (chapter 6 of [CS06]).

Main goal II: further topics

Talks 10-12 (Haoran, Wenhan) Following Shahidi's notes for AWS 2018, understand some key ingredients of Coates–Wiles' original proof of the Iwasawa Main Conjecture.

References

[Bel09] Bellaïche J. *An introduction to the conjecture of Bloch and Kato*. Lectures at the Clay Mathematical Institute summer School, Honolulu, Hawaii, 2009.

[Bel21] Bellaïche J. *The Eigenbook: Eigenvarieties, families of Galois representations, p-adic L-functions*. Springer Nature, 2021.

[Col04] Colmez P. *Fontaine's rings and p-adic L-functions*. Lecture notes, 2004.

[CS06] Coates J, Sujatha R. *Cyclotomic fields and zeta values*. Berlin: Springer, 2006.

[Kat07] Kato K. *Iwasawa theory and generalizations*. International congress of mathematicians. 2007, 1: 335-357.

[Och23] Ochiai T. *Iwasawa Theory and Its Perspective*, Volume 1. American Mathematical Society, 2023.

[Och24] Ochiai T. *Iwasawa Theory and Its Perspective*, Volume 2. American Mathematical Society, 2024.

[Ski09] Skinner C. *Galois representations, Iwasawa theory and special values of L-functions*. Lectures at the Clay Mathematical Institute summer School, Honolulu, Hawaii, 2009.

[Tat84] Tate J. *Les conjectures de Stark sur les fonctions L d'Artin en* $s = 0$*: notes d'un cours à Orsay rédigées par Dominique Bernardi et Norbert Schappacher*. Birkhäuser, 1984.

[Was21] Washington C. *Introduction to cyclotomic fields*. Springer Science & Business Media, 2012.