Higher Coleman Theory Study Group

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In the 1990s, Coleman proved his celebrated *classicality theorem* for overconvergent modular forms of small slope in [Col96], using a cohomological approach that crucially relied on the geometry of the modular curve. While Coleman's method is highly explicit, its geometric nature makes it difficult to generalize beyond the modular curve case. Remarkably, about a decade ago, Tian and Xiao achieved what seems to be the only known generalization of Coleman's approach, extending it to Hilbert modular varieties in [TX16b]. At the heart of their argument lies their previous work [TX16a], which provides a detailed understanding of the geometry and Goren–Oort stratification of Hilbert modular varieties.

On the other hand, a different approach was initiated by Kassaei [Kas06], building on earlier observations of Buzzard and Taylor [BT99]. Very roughly speaking, the idea was to analytically continue and glue overconvergent modular forms of small slope to the entire modular curve, and then apply rigid-analytic GAGA to conclude that these forms are classical. We shall informally refer to here this line of reasoning as the "GAGA approach". Thanks to the work of many authors (see e.g. [BPS16]), the GAGA approach has been generalized to a wide class of Shimura varieties.

The aim of this study group is to understand yet another approach to classicality, namely the Higher Coleman theory, developed by Boxer and Pilloni in [BP22] for the modular curve case, and in [BP21] for Shimura varieties of abelian type. The new ingredients are the use of local coherent cohomologies on Shimura varieties coming from pullbacks of the "Bruhat cells" on the flag varieties via the Hodge–Tate period map, together with lower bounds for the slopes of certain Hecke operators acting on these local cohomologies. From these ingredients, Boxer and Pilloni deduced classicality theorems via a spectral sequence associated to a filtration on the Shimura variey, whose E_1-page consists of local cohomology groups and which converges to coherent cohomology of the Shimura variety. One of the main advantages of the method is that it generalizes Coleman's theorem to higher coherent cohomologies of Shimura varieties.

The plan of the study group is to closely follow Boxer and Pilloni's paper [BP21, BP22], and also their lecture notes [BP20] on the same subject. However, rather than working through the group theoretic arguments in full generality, we hope to emphasize concrete computations and examples, focusing primarily on the cases of modular curve, Siegel threefold, and (if time permits) the Hilbert modular varieties. In particular, one interesting direction is to investigate how Higher Coleman theory may shed new light on Tian–Xiao's work, especially in the boundary case.

Talk 1-Coleman's proof of the classicality theorem
Explain the idea in [Col96, Proposition 6.6], which shows classicality for small slopes, and also [Col96, Lemma 7.3] for the boundary case, which deduces a classicality condition for overconvergent Hecke eigenform of critical slope.

Talk 2-Higher Coleman theory for the modular curve I
Following [BP22, Section 5], introduce cohomology with support, the canonical subgroup, the degree function, U_p-operator and its action on cohomologies. Carefully explain the proof of [BP22, Proposition 5.8].

Talk 3-Higher Coleman theory for the modular curve II
Explain the proof of [BP22, Lemma 5.11], and deduce the classicality theorem [BP22, Theorem5.13]. Following [BP22, Section 5.5], construct p-adic interpolation of sheaf, cohomology and show that the U_p-operator acts on the cohomologies.

Talk 4-Overview of the Higher Coleman theory
Following [BP20, Section 1.1], explain the construction of the Cousin complex for the flag variety $B \setminus G$, and compute the $\mathbf{P}^1$ case in detail. Give a brief overview of the theory, in particular state the target vanishing theorems and classicality theorems. The main reference is [BP20, Lecture 1]. Also check the notes [Hof20], which gives a very nice overview of the theory and specialized to the GSp_4 case.

Talk 5-Shimura varieties, automorphic vector bundles and action of the Hecke algebra
Follow [BP21, §4.1-4.3]. Do explicit computations in the GSp_4 case, see for example [Ort24, §1].

Talk 6-The Hodge Tate period map
Following [BP21, §4.4], explain the Hodge Tate period map in the Siegel case. Also check the original paper [Sch15, Chapter 3].

Talk 7-Toolkit in local cohomology theory
Follow [BP21, §2].

Talk 8-Flag varieties and dynamics of the torus action
Follow [BP21, §3]. Identify the Hecke operators in the Hecke algebra $\mathcal{H}^+_{K_p}$ in the GSp_4 case.

Talk 9-Overconvergent Cohomologies
Following [BP21, §5.1-5.5], explain the definition of the overconvergent coho-

mologies, the spectral sequence, and the Cousin complex associated to the Bruhat stratification. Compute the GSp_4 case in details.

Talk 10-Cohomlogical vanishing and lower bounds on slpoes
Carefully explain the proof of [BP21, Theorem 5.6.1] and [BP21, Theorem 5.9.6]. If time permits, talk about slope bounds in classical coherent cohomology in [BP21, Theorem 5.10]. Compute the GSp_4 case in details.

Talk 11-Slope conditions and the classicality theorem
Follow [BP21, §5.11, §5.12.1]. Compute the GSp_4 case in details.

Talk 12-BGG resolution
Explain Faltings' BGG resolution, compute the $\mathbf{GL}_2$, $\mathbf{GSp}_4$ and Hilbert case. References to be added.

Talk 13-Vanishing theorems for Betti and De Rham cohomology
Follow [BP21, §5.12.10, §5.13].

Talk 14-The Hilbert case
For the last talk, we run the computations of higher Coleman theory for the Hilbert case as an exercise, and compare the results we get with [TX16b].

References

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