Remarks on last time

- $\pi_n : X_n \to X$, $E_n/\mathbb{Q}_p$ unique unr. extn. of deg n ; $X_n := X \times_{\mathbb{Q}_p} E_n$
- $\mathcal{O}(d,n) := \pi_{n,*}(\mathcal{O}_{X_n}(d))$

e.g. $\mathcal{O}(2,2) \simeq \mathcal{O}(1)^2$

In general, "$\mathcal{O}(d,n) = \mathcal{O}(d/n)^{\oplus \delta}$", where $\delta = (d,n)$.

2) $\pi_{m,*}(\mathcal{O}_{X_m}(d,n)) = \mathcal{O}_X(d,nm)$; $\pi_m^*(\mathcal{O}_X(d,n)) = \mathcal{O}_{X_m}(md,n)$

3) $\mathcal{O}_X(d,n)$ is semi-stable of slope d/n

4) $\mathcal{O}_X(d_1,n_1) \otimes \mathcal{O}_X(d_2,n_2) = \mathcal{O}_X(d_1n_2 + d_2n_1, n_1n_2)$;

$\mathcal{O}_X(d,n)^\vee = \mathcal{O}_X(-d,n)$

Pf (sketch): 1) & 2) follows from the projection formula: e.g. if $d = d_1 n$, then

$$\mathcal{O}_X(d,n) = \pi_{n,*}(\mathcal{O}_{X_n}(d)) = \pi_{n,*}(\pi_n^* \mathcal{O}_X(d_1))$$
$$= \mathcal{O}_X(d_1) \otimes \pi_{n,*}\pi_n^*(\mathcal{O}_X) = \mathcal{O}_X(d_1) \otimes \mathcal{O}_X^n = \mathcal{O}_X(d_1)^n.$$

3) follows from 1) & 2).: Since π_n^* multiplies the slope of any vb. by n, ISTS: $\pi_n^*(\mathcal{O}_X(d,n)) = \mathcal{O}_{X_n}(dn,n) = \mathcal{O}_{X_n}(d)^n$ is semi-stable.

Fact: On any curve an extn of semi-stab. vb of the same slope is again semi-stab.

* Prop 8, Lec 20. [Lurie]

4) WLOG, $(n_1, n_2) = 1$. Look at

$$\begin{tikzcd} & X_{n_1 n_2} \arrow[dl, dash] \arrow[dr, dash] & \\ X_{n_1} \arrow[dr, dash] & & X_{n_2} \arrow[dl, dash] \\ & X & \end{tikzcd}$$

and pull back every bdle to $X_{n_1 n_2}$. □

- $f : X \to Y$ fin. flat, deg n then

$f^* f_* \mathcal{O}_X \simeq \mathcal{O}_X^n$ if f is a fin. ét. cover.

$f_* f^* \mathcal{O}_Y \simeq f_* \mathcal{O}_X$ a vb on Y, but almost never $\simeq \mathcal{O}_Y^n$ (true if $X \to Y$ is the triv. cover of deg n)

- e.g. $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ affine $f^* f_* \mathcal{O}_X \leftrightsquigarrow B \otimes_A B$; $f_* f^* \mathcal{O}_Y \leftrightsquigarrow B$

Lem: $\mathcal{E}, \mathcal{E}'$ vbs on X semi-stab $\Rightarrow$ $\mathcal{E} \otimes \mathcal{E}'$ is semi-stab.
↳ assuming the classification thm.

- $B \otimes_A B \simeq B^n$ if B/A is Galois, in which case $B \otimes_A B \simeq \bigoplus_{\sigma \in Gal(B/A)} B^\sigma$

§4. Application: weakly admissible $\Rightarrow$ admissible.

- $K = \mathbb{Q}_p$
- $G_{\mathbb{Q}_p} = Gal(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)$
- $Rep(G_{\mathbb{Q}_p})$:= cat. of cts repns of $G_{\mathbb{Q}_p}$ on fin. dim'l $\mathbb{Q}_p$-v.s.

Fontaine:

Hodge – Tate
$\cup$
B_{dR}-adm. = de Rham
$\cup$
pot. semi-stab.
$\cup$
semi-stab.
$\cup$
B_{crys}-adm. = crystalline

defined in terms of period rings $\rightsquigarrow$ "different type of redn on the geom. side of the picture"

- B (large) $\mathbb{Q}_p$-alg. $\circlearrowleft G_{\mathbb{Q}_p}$
- $F = B^{G_{\mathbb{Q}_p}}$
- $D_B(V) := (B \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$
- adm. if $\dim_F D_B(V) = \dim_{\mathbb{Q}_p} V$

$Rep_B(\mathbb{Q}_p) \xrightarrow{D_B} \{\text{fin. dim. v.s.} / F\}$ (+ add'l str., inherited from those of B.)

hope: fully faithful

de Rham (resp. crys.) comes from ét. coh. of sm. p-adic vars (resp. of good redn)

Taking both filn. & Frobenius into account

- φ-$ModFil_{\mathbb{Q}_p}$: $(\underbrace{D, \varphi_D}_{\in\ \mathbb{F}_p\text{-isocrystal}}, Fil^\bullet D)$ & sep. + exhaustive : $\cap_i Fil^i = \{0\}$, $\cup_i Fil^i = D$

  $(D, Fil^\bullet D) \in VectFil_{\mathbb{Q}_p}$

  V_{crys}

  $\Downarrow$

  HN formalism applies

  rk := $\dim_{\mathbb{Q}_p} D$

  $deg = deg(D, \varphi_D) + \underbrace{deg(D, Fil^\bullet D)}_{= \sum_i i \dim_{\mathbb{Q}_p}(gr^i)}$

- D_{crys} : $Rep_{crys}(G_{\mathbb{Q}_p}) \longrightarrow \varphi$-$ModFil_{\mathbb{Q}_p}$ is fully faithful
- Fontaine conjectured : A filtered isocrystal is in the essential image of D_{crys} iff it is weakly adm. $\Leftrightarrow$ "semi-stab. of slope zero."

"$\Rightarrow$" easy "$\Leftarrow$" Colmez – Fontaine 2000 ; Berger 2008

Thm: Fontaine's conj is true.

Pf (assuming clsn thm of vbs on the FF curve):

Given a filtered isocrystal $(D, \varphi_D, \mathrm{Fil}^\bullet D) \in \varphi\text{-Mod}_{\mathbb{Q}_p}$,

$$((B_{crys} \otimes_{\mathbb{Q}_p} D)^{\varphi=1}, \; \mathrm{Fil}^0(B_{dR} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p))$$

defines a v.b. on X "(B_{dR}, ν)-pair", say $\mathcal{E}$

⌜ The idea comes from $\mathbb{P}^1$:

$$\mathrm{Bun}(\mathbb{P}^1_{\mathbb{C}}) \xrightarrow{\sim} \{\text{pairs } (M, M_\infty) : M \text{ fin. free } \mathbb{C}[z]\text{-mod.}, \; M_\infty \; \mathbb{C}[\![z_\infty]\!]\text{-lattice inside } M \otimes_{\mathbb{C}[z]} \mathbb{C}(\!(z_\infty)\!)\}$$

$\mathbb{A}^1_{\mathbb{C}}$ ↑, ↑ infinitesimal nbhd of ∞ $(z_\infty = 1/z)$ ⌟

Now, suppose D is semi-stab. of slope zero. Then so is $\mathcal{E}$. ← not completely automatic from HN, but not hard to show.

$$H^0(X, \mathcal{E}) = (B_{crys} \otimes_{\mathbb{Q}_p} D)^{\varphi=1} \cap \mathrm{Fil}^0(\cdots)$$
$$= V_{crys}(D)$$

coh. interpretation of $V_{crys}(D)$ ↑ left inverse functor of D_{crys}

$\Rightarrow \mathcal{E}$ is triv., i.e., $H^0(X, \mathcal{E}) \otimes \mathcal{O}_X \simeq \mathcal{E}$

$\Rightarrow \dim_{\mathbb{Q}_p} V_{crys}(D) = \dim_{\mathbb{Q}_p} D \rightsquigarrow D = D_{crys}(V_{crys}(D))$. □