Last time

(Dieudonné - Manin)

- The cat. of isocrystals / $\bar{\mathbb{F}}_p$ is semi-simple w/ simple objs $D(\lambda)$, $\lambda \in \mathbb{Q}$

  ⌜ $\breve{\mathbb{Q}}_p := W(\bar{\mathbb{F}}_p)[1/p]$, φ = lift of $\varphi_{\bar{\mathbb{F}}_p}$

  $M \in \mathrm{Isoc}(\bar{\mathbb{F}}_p) :=$ $\breve{\mathbb{Q}}_p$ - mod. w/ $\varphi_M : \varphi^* M \xrightarrow{\sim} M$

  $D(d/n) = \breve{\mathbb{Q}}_p{}^n$, $\varphi_\lambda \leftrightarrow \begin{pmatrix} 0 & \cdots & p^d \\ 1 & 0 & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots 1 & 0 \end{pmatrix}$ ⌟

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- Main goal: $\mathrm{Bun}_{X^{FF}} \leftrightsquigarrow \mathrm{Isoc}(\bar{\mathbb{F}}_p)$

  bij. on objs, NOT equiv of cats

  - Harder - Narasimhan formalism applies to $\mathrm{Isoc}(\bar{\mathbb{F}}_p)$ ($rk = \dim_{\breve{\mathbb{Q}}_p} D$, $\deg D = -\nu(\det(\varphi_D))$)

  - Classification Theorem: Let $\mathcal{E}$ be a vb on X^{FF}. Then $\exists !$ seq. of rat'l numbers $\lambda_1 \geq \cdots \geq \lambda_m$ s.t.

$$\mathcal{E} \simeq \bigoplus_{i=1}^{m} \mathcal{O}_{X^{FF}}(\lambda_i).$$

  - Functor $\mathcal{E}(-) : \mathrm{Isoc}(\bar{\mathbb{F}}_p) \to \mathrm{Bun}_{X^{FF}}$

$$D \longmapsto \pi_{n,*}(\mathcal{O}_{X^{FF}_{E_n}}(d))$$

$$X^{FF}_{E_n} \simeq X^{FF} \times_{\mathbb{Q}_p} E_n \xrightarrow{\pi_n} X^{FF}$$

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  $E_n / \mathbb{Q}_p$ unr. of deg n

$$X^{FF} = \mathrm{Proj}\Big(\bigoplus_{k\geq 0} B^{\varphi = p^k}\Big) \qquad X^{FF}_{E_n} = \mathrm{Proj}\Big(\bigoplus_{k\geq 0} B^{\varphi^n = p^k}\Big)$$

$$\mathcal{O}(d) := \widetilde{\bigoplus_{k\geq 0} B^{\varphi = p^{k+d}}}$$

$$X^{FF}_{E_n} \simeq \mathrm{Proj}\Big(\underbrace{\bigoplus_{k\geq 0} B^{\varphi^n = p^{kn}}}\Big) \qquad \mathrm{Gal}(E_n/\mathbb{Q}_p) \simeq \langle \psi \rangle$$

E_n-v.s. $\circlearrowleft$ $\psi = p^{-k}\varphi$ loc. fin.

Hilbert 90 $\Rightarrow B^{\varphi^n = p^{kn}} \simeq B^{\varphi = p^k} \otimes E_n$

  - First step: $\mathrm{Pic}(X^{FF}) \simeq \mathbb{Z}$, via $\mathrm{Div}(X^{FF}) \to \mathrm{Pic}(X^{FF})$ and $\deg$

    - inj. $\forall x \in X^{FF}$, $\mathcal{O}(x) \simeq \mathcal{O}(1)$ 1
    - surj. $H^0(X^{FF}, \mathcal{O}(1)) \xrightarrow{\sim} B^{\varphi = p}$

- More gen., $H^i(X^{FF}, \mathcal{O}_X^{FF}(d)) = \begin{cases} B^{\varphi=p^d}, & i=0 \\ 0 & i=2, \text{ or } i=1 \ \& \ n\geq 0 \\ B^+_{dR,x}/\mathrm{Fil}^{-n}B^+_{dR,x} + \mathbb{Q}_p \neq 0, & i=1 \ \& \ n\leq -1 \end{cases}$

Today (+ next week)

① Compute coh. of line bdls on X^{FF}

② Study covers of X^{FF} & show that $\pi_1^{ét}(X^{FF}) \simeq \mathrm{Gal}_{\mathbb{Q}_p}$

③ Sketch some key ideas in the proof of the classification theorem

④ weakly adm. $\Rightarrow$ adm.

§1. Cohomology of $\mathcal{O}(d)$ on X^{FF}

! Assume C^b alg cl. throughout

Recall: $P := \bigoplus_{k\geq 0} B^{\varphi=p^k}$ is graded factorial w/ irred elts in deg 1.

Lem: $H^0(X^{FF}, \mathcal{O}_X(d)) \simeq B^{\varphi=p^d}$.

Pf: Since $\mathcal{O}_X(d) = \widetilde{\bigoplus_{k\geq 0} B^{\varphi=p^{k+d}}}$, $\exists$ nat'l map

$$B^{\varphi=p^d} \longrightarrow H^0(X^{FF}, \mathcal{O}_{X^{FF}}(d)) \ni s$$

Inj. because P is factorial.

surj. same proof as in $\mathbb{P}^n$ case. (cover X^{FF} by $D_+(f) = ((P)_f)_0$ w/ f irred.; a section s becomes an elt. in $((P)_f)_d$)

□

Prop: $H^i(X^{FF}, \mathcal{O}_{X^{FF}}(d))$ is computed as above.

Pf: $i=0$ ✓ $i\geq 2$ follows from the fact that X^{FF} can be covered by two open affines. (Pick 2 non-colinear $t, t' \in B^{\varphi=p}$, then $X^{FF} = \mathrm{Spec}(B_t) \cup \mathrm{Spec}(B_{t'})$. Here, $B_t = ((P)_t)_0$ is a PID [Thm 10.6, Ans] & we've seen that $X = \mathrm{Spec}(B_t) \cup \{\infty_t\}$ & $X = D_+(t) \cup V_+(t)$, $V_+(t) = \mathrm{Proj}(P/tP) = \{pt\}$ [Cor 10.3, Ans])

$i=1$. Consider the SES ($m\geq 0$)

skyscraper sheaf at ∞_t

$$0 \longrightarrow \mathcal{O}_X(m) \xrightarrow{t} \mathcal{O}_X(m+1) \longrightarrow k(\infty_t) \longrightarrow 0$$

(this is analogous to $\mathbb{P}^1$; look at the stalks to check this is indeed an SES)

LES
$\Rightarrow 0 \to H^0(X^{FF}, \mathcal{O}_{X^{FF}}(m)) \to H^0(X^{FF}, \mathcal{O}_{X^{FF}}(m+1)) \to k(\infty_t) \xrightarrow{\delta}$

$\to H^1(X^{FF}, \mathcal{O}_{X^{FF}}(m)) \to H^1(X^{FF}, \mathcal{O}_{X^{FF}}(m+1)) \to 0 \to 0 \cdots$

Key: $\delta = 0$ ("the fundamental exact sequence in p-adic Hodge theory")

Given this,

$$H^1(X^{FF}, \mathcal{O}_{X^{FF}}) \xrightarrow{\sim} \varprojlim_m H^1(X^{FF}, \mathcal{O}_{X^{FF}}(m)) \simeq H^1(X^{FF}, \underbrace{\varprojlim_m \mathcal{O}_{X^{FF}}(m)}_{\simeq j_* \mathcal{O}_X \text{ where}})$$

$$\simeq H^1(\mathrm{Spec}(B_t), \mathcal{O}_{\mathrm{Spec}(B_t)}) = 0. \quad j: \mathrm{Spec}(B_t) \hookrightarrow X^{FF}$$

Now, it remains to show that the seq.

untilt corresp. to $x \in X^{FF}$

$$0 \to B^{\varphi = p^m} \xrightarrow{\cdot t} B^{\varphi = p^{m+1}} \to B^+_{dR,x}/\xi_x \simeq K_x \to 0$$

Note that, $B^{\varphi=p}/tB^{\varphi=1} \longrightarrow K_x$

for any $s \in B^{\varphi=p}$ not colinear w/ t. $\quad \downarrow \cdot s^m \quad ? \quad s\downarrow \cdot s^m \quad \Rightarrow$ reduce to the case $m=0$

$B^{\varphi=p^{m+1}}/tB^{\varphi=p^m} \longrightarrow K_x$ $\quad$ which is something we've seen:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Q}_p & \to & B^{\varphi=p} & \xrightarrow{\theta_x} & K_x & \to & 0 \\ & & \uparrow & & \wr\uparrow \log & & & & \\ 0 & \to & \varepsilon^{\mathbb{Q}_p} & \to & 1+\mathfrak{m}_{C^\flat} & \to & K_x & \to & 0 \end{array}$$

(over $E/\mathbb{Q}_p$ fin., needs Lubin-Tate thy)

For $m \leq -1$, consider

$$0 \to \mathcal{O}_X^{FF}(-1) \xrightarrow{\cdot t} \mathcal{O}_X^{FF} \to k(\infty_t) \to 0$$

LES
$\Rightarrow H^1(X^{FF}, \mathcal{O}_X^{FF}(-1)) \xrightarrow{\sim} K_x/\mathbb{Q}_p$ + induction on m. $\square$

Rmk: 1) $H^1(X^{FF}, \mathcal{O}_X(m)) = 0$ for $m \geq 0$ is basically equiv. to the FES. [Lur] Lec 19 established it via a cute coh. computation.

2) X complete alg. curve / alg. cl. field k. Then X has genus $0 \Leftrightarrow \mathrm{Pic}(X) \simeq \mathbb{Z}$ & $H^1(X, \mathcal{O}_X) = 0$. $\rightsquigarrow X^{FF}$ behaves like a curve of genus 0.

§2. Covers

- $E/\mathbb{Q}_p$ fin. extn of deg n
- $X_E^{FF} := X^{FF} \times_{\mathbb{Q}_p} E$

First properties:

1) $\pi: X_E^{FF} \to X^{FF}$ is fin. ét. since $\mathrm{Spec}(E) \to \mathrm{Spec}(\mathbb{Q}_p)$ is.

2) Since $\mathbb{Q}_p \xrightarrow{\sim} H^0(X^{FF}, \mathcal{O}_{X^{FF}})$, $\underset{\varphi=1}{E} \xrightarrow{\sim} H^0(X_E^{FF}, \mathcal{O}_{X_E^{FF}}) \Rightarrow X_E$ is conn.

3) X^{FF} is covered by $\underbrace{\mathrm{Spec}(B[1/t])}_{=:U}$ for $t \in \bigoplus_{k>0} B^{\varphi=p^k}$ homogeneous $\rightsquigarrow U_E := U \times_{\mathbb{Q}_p} E$, which can be described by

$$B[1/t]^{\varphi=1} \otimes_{\mathbb{Q}_p} E \simeq (B[1/t] \otimes_{\mathbb{Q}_p} E)^{\varphi=1}$$

covers X_E^{FF}. $\uparrow$ φ acts triv. on E

4) $x \in X^{FF}$ cl. pt. then $X_E^{FF} \times_{X^{FF}} \{x\} \simeq \mathrm{Spec}(E \otimes_{\mathbb{Q}_p} K_x) = \bigsqcup_n \mathrm{Spec}(K_x) \rightsquigarrow$ every cl. pt. of X^{FF} has exactly n pts of X_E lying above it, each having the same res. field.

5) $|X_E^{FF}| \xleftrightarrow{1-1} \{\underbrace{K, \iota}_{\text{untilt}}, u\}$, where $u: E \to K$ is a map of $\mathbb{Q}_p$-alg.

e.g. $E = E_n$ unr. of deg n. then $E = W(\mathbb{F}_{p^n})[1/p]$ &

$$E \to K \leftrightarrow W(\mathbb{F}_{p^n}) \to \mathcal{O}_K \leftrightarrow \mathbb{F}_{p^n} \to \mathcal{O}_K / p\mathcal{O}_K \leftrightarrow \mathbb{F}_{p^n} \to \mathcal{O}_{C^\flat}/\varpi$$

$$\leftrightarrow \mathbb{F}_{p^n} \to C^\flat$$

n elts cyclically permuted by $\varphi_{C^\flat}$

$$\Rightarrow |X_E^{FF}| \simeq |Y_E| / \varphi^{\mathbb{Z}} \simeq |Y \times \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^n}, C^\flat)| / \varphi^{\mathbb{Z}} \simeq |Y| / \varphi^{n\mathbb{Z}}.$$

Thm: $\mathrm{Pic}(X_E^{FF}) \simeq \mathbb{Z}$.

Pf: See [Lur], Lec. 22-25. "Lubin-Tate theory". $\square$

Thm: X^{FF} is geom. simply conn., i.e., the projection map $X \to \mathrm{Spec}(\mathbb{Q}_p)$ induces an iso. of ét. fund. gps $\pi_1^{\text{ét}}(X^{FF}) \xrightarrow{\sim} \pi_1^{\text{ét}}(\mathrm{Spec}\,\mathbb{Q}_p) \simeq \mathrm{Gal}_{\mathbb{Q}_p}$. Equiv., pullback along $X^{FF} \to \mathrm{Spec}(\mathbb{Q}_p)$ induces an equiv. of cats

$$\{\text{Ét. covers of } \mathrm{Spec}(\mathbb{Q}_p)\} \xrightarrow{\sim} \{ -''- \ X^{FF} \}.$$

Lem: $\mathcal{E}, \mathcal{E}'$ vbs on X^{FF} semi-stab. of slope μ, μ', resp. Then $\mathcal{E} \otimes \mathcal{E}'$ is semi-stab. of slope $\mu + \mu'$.

preserve semi-stab

Pf: WLOG. $\mathcal{E} = \pi_* \mathcal{O}_{X_E^{FF}}(d) \Rightarrow \mathcal{E} \otimes \mathcal{E}' = \underset{(4)}{\pi_*}(\mathcal{O}_{X_E}(d) \otimes \pi^* \mathcal{E}')$. $\square$

Proof of Thm.

Let $\pi : \tilde{X} \to X$ be a fin. ét. cover. WTS: $\tilde{X} = X \times_{\mathbb{Q}_p} \overset{\text{ét. } \mathbb{Q}_p\text{-alg}}{\overset{\shortparallel}{E}}$. prod. of fin. extn.

Put $\mathcal{E} := \pi_* \mathcal{O}_{\tilde{X}}$ & define $E := H^0(X, \mathcal{E})$. ISTS: $\mathcal{E} \simeq E \otimes_{\mathbb{Q}_p} \mathcal{O}_X$, i.e., $\mathcal{E}$ is triv. $\underset{\text{Thm}}{\overset{\text{Clsn}}{\Longleftarrow}}$ $\mathcal{E}$ is semi-stab. of slope 0.

- π fin. ét. $\Rightarrow$ $\mathcal{E} \otimes \mathcal{E} \to \mathcal{E} \xrightarrow{\text{tr}} \mathcal{O}_X$ non-degen. $\Rightarrow \mathcal{E} \simeq \mathcal{E}^\vee$. But $\deg(\mathcal{E}) = -\deg(\mathcal{E}^\vee) \Rightarrow \deg(\mathcal{E}) = 0 \Rightarrow$ slope 0 ✓
- Suppose $\mathcal{E}$ is not semi-stab. Let $\mathcal{E}' \subset \mathcal{E}$ be the first step of the HN filn. of $\mathcal{E}$. Then $\mathcal{E}'$ is semi-stab. w/ slope $\mu > 0$ & $\forall\, \mathcal{F}$ s.s. v.b. of slope $> \mu$, $\mathcal{F} \to \mathcal{E}$ is triv.

  ⌜ $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$ w/ $\mathcal{E}_i / \mathcal{E}_{i-1}$ s.s. & $\mu_i := \mu(\mathcal{E}_i / \mathcal{E}_{i-1})$ is descending strictly. ⌟

  Lem $\Rightarrow \mathcal{E}' \otimes \mathcal{E}' \hookrightarrow \mathcal{E} \otimes \mathcal{E} \xrightarrow{m} \mathcal{E}$ is triv. But $\exists$ an aff. open $U \subset X$ over which $\mathcal{E}'$ has a non-zero section s, & can view s as a non-zero reg. fcn. on $\underbrace{\tilde{X} \times_X U}_{\text{red.}}$ w/ $s^2 = 0$ #.

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Aside: $\pi_1^{\text{ét}}(\operatorname{Spec} \mathbb{Q}_p)$ $\Rightarrow$ {ét. loc. systems on X} $\simeq$ {fin. ab gp. w/ $\mathrm{Gal}_{\mathbb{Q}_p}$-action}

- Choose a geom. pt. $\bar{x} \to \operatorname{Spec} \mathbb{Q}_p$ $\leftrightarrow$ choosing an alg. cl. $\mathbb{Q}_p \to \overline{\mathbb{Q}}_p$
- Consider the fiber functor

$$F_{\bar{x}} : \mathrm{FÉt}(\operatorname{Spec} \mathbb{Q}_p) \to \text{Sets}$$
$$E \mapsto \operatorname{Hom}_{\mathbb{Q}_p\text{-Alg}}(E, \overline{\mathbb{Q}}_p)$$

- $\pi_1^{\text{ét}}(\operatorname{Spec} \mathbb{Q}_p) := \operatorname{Aut}(F_{\bar{x}}) \simeq \mathrm{Gal}_{\mathbb{Q}_p}$.

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§ 3. The Classification Thm.

Criterion (X): For every v.b. $\mathcal{E}$ on X & for every $n \geq 1$, if $\exists$ SES

$$0 \to \mathcal{O}_X(-1/n) \to \mathcal{E} \to \mathcal{O}_X(1) \to 0$$

then $H^0(X, \mathcal{E}) \neq 0$.

Thm: Suppose criterion (X_n) holds for every cyclic unr. covering X_n of deg n, $\forall n$; then

i) The semi-stab. vbs on X are isoclinic (pure)

ii) The HN filn always splits

iii) Every vb. on X is iso. to $\bigoplus_{i=1}^{m} \mathcal{O}_X(\lambda_i)$ for unique rat'l numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$.

(Furthermore, if $C(X_n)$ holds only for vb.s of rk $\leq r$, $\forall n$, then i) – iii) ✓ for vb. of rk $\leq r$.)

Pf skh: Observe i) $\Rightarrow$ ii) $\Rightarrow$ iii)

$$\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = H^1(X, \mathcal{O}(\mu-\lambda)) = 0 \text{ if } \mu \geq \lambda$$

Toy case: $0 \to \mathcal{E}_1 \to \mathcal{E} \to \underbrace{\mathcal{E}/\mathcal{E}_1}_{\mu_2} \to 0$, $\quad \mu_1 > \mu_2$

$\Rightarrow$ splits. ✓ In gen., induction on length, rk. ⋯

Lem 76, [Shalit]

Step 1. $\mathcal{E}$ is s.s. $\Leftrightarrow$ $\pi^*\mathcal{E}$ is, so WLOG pull back to X_n and assume that $\mu(\mathcal{E}) \in \mathbb{Z}$. Twisting by $\mathcal{O}(m)$ remains s.s. $\Rightarrow$ WLOG, $\mu(\mathcal{E}) = 0$.

ISTS: If $\mathcal{E}/X_n$ is s.s. of slope 0, then $\mathcal{E}$ is triv.

Induction on rk$(\mathcal{E})$. Assume ✓ for rk $\leq r$.

Step 2. Write rk$(\mathcal{E}) = r+1$, consider $\pi_r^*(\mathcal{E})$. Let $\mathcal{L} \subset \pi_r^*(\mathcal{E})$ be a line bdl of max. deg d. Consider SES

$$0 \to \mathcal{L} \to \pi_r^*(\mathcal{E}) \to \overset{\text{s.s.}}{\mathcal{E}'} \to 0 \qquad (*)$$

$\Rightarrow d \leq 0 \leq \mu(\mathcal{E}')$.

Step 3. $\mathcal{E}$ is triv. if $d = 0$. Indeed, $\mathcal{L}$ s.s. of slope 0. But the cat. of ss. vb. of slope 0 is ab. $\Rightarrow$ so is $\mathcal{E}'$. Induction on rk. $\Rightarrow$ $\mathcal{E}'$ is triv. $\Rightarrow$ $\mathcal{E}$ is triv. since $\mathrm{Ext}^1(\mathcal{O}^r, \mathcal{O}) = 0$.

Step 4. It is impossible to have $d \leq -2$.

Claim: $\exists \lambda \geq 0$ w/ $\mathcal{O}_{X_r}(\lambda) \hookrightarrow \mathcal{E}'$. (induction on rk.). $d \leq -2 \Rightarrow \exists$ non-triv. hom. $\mathcal{O}_{X_r}(d+2) \to \mathcal{O}_{X_r}(\lambda) \subset \mathcal{E}'$. Pull back SES $(*)$ $\Rightarrow$ SES

$$0 \to \mathcal{L} \to \mathcal{E}'' \to \mathcal{O}_{X_r}(d+2) \to 0$$

$\Leftrightarrow$

$$0 \to \underbrace{\mathcal{L}(-d-1)}_{\text{is } \mathcal{O}_{X_n}(-1)} \to \mathcal{E}''(-d-1) \to \mathcal{O}_{X_r}(1) \to 0$$

Criterion $(X_r) \Rightarrow H^0(X_r, \mathcal{E}''(-d-1)) \neq 0 \Leftrightarrow \exists$ non-triv. hom.

$$\mathcal{O}_{X_r}(d+1) \to \mathcal{E}''.$$

Since $\mathcal{E}'' \to \pi_r^*(\mathcal{E})$ is mono., we get a non-triv. $\mathcal{O}_{X_r}(d+1) \xrightarrow{v} \pi_r^*(\mathcal{E})$. The line sub-bdl spanned by the image of v has deg $\geq d+1$. #.

Step 5 It is impossible to have $d=-1$.

See e.g. [Sha], Thm 7.9.

$\square$

Big picture.

Reduction to deg 1 modifications of vbs

$C'(X)$: 1) If $\mathcal{O}_X(1/n)$ is an increasing modn of $\mathcal{E}$ of deg 1, i.e., $\exists$ SES

$\leftrightarrow$ 1-dim formal LT gp of ht n

$$0 \to \mathcal{E} \to \mathcal{O}_X(1/n) \to \mathcal{F} \to 0$$

w/ $\mathcal{F} = i_{x,*}k(x)$ for a cl. pt. x, then $\mathcal{E} \simeq \mathcal{O}_X^n$.

2) If $\mathcal{E}$ is an increasing modn. of $\mathcal{O}_X^n$ of deg 1, i.e.,

$$0 \to \mathcal{E} \to \mathcal{O}_X^n \to \mathcal{F} \to 0$$

w —"—, then $\mathcal{E} \simeq \mathcal{O}_X^{n-m} \oplus \mathcal{O}_X(-1/m)$ for some $m \in \{1, \cdots, n\}$.

Thm: $C'(X) \Leftrightarrow C(X)$. ⚠ The author uses a different variant of 2) there.

Pf: See e.g. Chao Li's notes "Vector bundles on the Fargues-Fontaine curve".

A modn of a vb at $x \leftrightarrow$ a B_{dR}^+-lattice inside the B_{dR}-v.s. assoc. to the vb.

deg 1 $\leftrightarrow$ "miniscule" condn

$\leadsto$ We have reduced a global problem on the curve to a local linear alg. problem over B_{dR}^+.

How do we classify all possible B_{dR}^+-lattices of $\mathcal{O}_X(1/n)$ & $\mathcal{O}_X^n$?

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period lattices of p-div. gps

$\leadsto$ Study period maps on certain Rapoport deformation space of p-div. gps.

1) Surj. of the G.-M. period map from the Lubin-Tate sp. to $\mathbb{P}^{n-1}$ (Gross-Hopkins)

2) The image of the de Rham period of the RZ sp. of special formal $\mathcal{O}_B$-mods of ht n^2 is exactly $\Omega \subset \mathbb{P}^{n-1}$ (Drinfeld)

1) $\mathcal{O}_X(1/n) = \mathcal{E}(D, \varphi)$ where D = isocrystal of the unique 1-dim. formal LT gp $\mathbb{H}$ of ht n / $\bar{\mathbb{F}}_p$

Min. modn $\leftrightarrow$ specific B_{dR}^+-lattice in $D \otimes B_{dR}^+$

$\leftrightarrow$ filn $\mathrm{Fil}\, D_{K_x}$ in $D_{K_x} = D \otimes_{\mathbb{Q}_p} K_x$ of dim $n-1$

$\leftrightarrow$ pt in $\mathbb{P}^{n-1}(K_x)$

$\uparrow \pi_{GM}$

RZ defn sp. $\to D(\mathcal{O}_{K_x})$

Upshot: Every min. modn. of $\hat{\mathcal{O}}_X(1/n)$ comes from a p-div. gp $G / \mathcal{O}_{K_x}$.

Scholze-Weinstein Thm: $\{$p-div. gp. $/ \mathcal{O}_{K_x}\} \xrightarrow{\sim} \{$min. modn. of triv. vbs on $X^{FF}\}$

$\Rightarrow \mathcal{E}$ is triv.

2) LES in coh. gives $0 \to H^0(X, \mathcal{E}) \to \underbrace{\mathbb{Q}_p^n \xrightarrow{ev} K_x}_{\text{defines a pt. } p \text{ in } \mathbb{P}^{n-1}_{K_x}} \to \cdots$

Claim: $H^0(X, \mathcal{E}) = 0 \iff \mathcal{E} \simeq \mathcal{O}_X(-1/n)$. Indeed, $H^0(X, \mathcal{E}) = 0 \Rightarrow ev$ is inj. $\iff p \in \Omega^{n-1}$:= open complement of all $\mathbb{Q}_p$-rat'l hyperplanes in the adic space assoc. to $\mathbb{P}^{n-1}_{K_x}$

Drinfeld: $\mathcal{M}(K_x) \xrightarrow{\pi_{dR}} \mathbb{P}^{n-1}(K_x)$ has image precisely Ω^{n-1}.

certain defn. sp. of p-div. gps w/ additional str.

$\overset{SW}{\Rightarrow} \mathcal{E} \simeq \mathcal{O}_X(1/n)$.

In general, $H^0(X, \mathcal{E}) \neq 0$ & so the pt lies on at least 1 $\mathbb{Q}_p$-rat'l hyperplane. Intersect all rat'l hyperplanes containing $p \rightsquigarrow$ get a min. rat'l subspace $V \subset \mathbb{Q}_p^n$ of dim. m.

$\rightsquigarrow p \subset \Omega^{m-1} \subset V \otimes K_x = K_x^m$ $\quad V^\perp$ untouched / triv. modn.

$\rightsquigarrow \mathcal{O}_X(-1/m) \quad\quad \rightsquigarrow \mathcal{O}_X^{n-m}$