Talk 5. Vector bundles on the Fargues - Fontaine Curve I.

Motivation : isocrystals

Def: Given (A, φ) w/ A ring & $\varphi \in \mathrm{End}(A)$. A φ-mod. is a fin. proj. A-mod. M w/ iso. $\varphi_M : \varphi^* M \xrightarrow{\sim} M$

e.g. $A = W(\overline{\mathbb{F}}_p)[1/p] = \breve{\mathbb{Q}}_p$ w/ $\varphi = \varphi_{\overline{\mathbb{F}}_p} = \mathrm{Frob}_p$. φ-mod. $/ (A, \varphi) =: \overline{\mathbb{F}}_p$ - isocrys.

1) 1-dim. case: $M = A$, $\varphi_M : 1 \mapsto p^m$ for $m \in \mathbb{Z}$

2) For $\lambda = d/r \in \mathbb{Q}$, $r > 0$, d, r coprime. Then

$$D(\lambda) = \breve{\mathbb{Q}}_p^{\oplus r}, \quad \varphi_\lambda \leftrightarrow \begin{pmatrix} 0 & \cdots & p^d \\ 1 & 0 & \\ \vdots & \ddots & \vdots \\ 0 & \cdots 1 & 0 \end{pmatrix}$$

Fact: The $D(\lambda)$ are all simple.

Thm (Dieudonné - Manin): The cat. of $\overline{\mathbb{F}}_p$ - isocrys. is semi-simple w/ simple obj $D(\lambda)$.

- These show up as rat'l crystalline coh. gps. e.g. $A / \overline{\mathbb{F}}_p$ ab. var., this is the "rat'l Tate module for $\ell = p$"
- These are the easiest part of the "semi-linear" machinery in p-adic Hodge theory.
- FF curve: It will "linearize" these (i.e. from s.l. alg. to l.a.)

Back to X_{FF}:

Fix $C^\flat$ alg. cl. perfectoid in char p. (we fix our base $\mathbb{Q}_p$), ϖ pseudo-uniformizer

Fact: 1) "$X_{FF}^{ad} \simeq \mathcal{Y}/\varphi^{\mathbb{Z}}$", $\mathcal{Y} = \mathrm{Spa}(\mathbb{A}_{\inf}) \setminus \{p[\varpi]\}$

2) $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) = B$

Recall: We constructed B as "holo. fcns" in the variable p on the punctured unit disc".
↑ this space is $\mathcal{Y}$

These motivates X_{FF}:

1) People knew $\left\{ \begin{array}{c} \text{cat. of p-adic Hodge} \\ \text{theoretic objs} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{vbs on } \mathbb{D}^\times + \text{Frob.} \\ \text{s.l. data extra stuff} \end{array} \right\}$

see e.g. Brinon - Conrad

② Want to quotient by φ. φ is not an automorphism on $\mathbb{D}^\times$

Let's construct vb. on X_{FF} from isocrystals!

We have $\mathcal{Y} \to \mathrm{Spa}\, \breve{\mathbb{Q}}_p \Rightarrow$ isocrystals can be pulled back to $\mathcal{Y}$ + descended to X_{FF}^{ad}! In particular, $\mathbb{Z} \hookrightarrow \mathrm{Pic}(X_{FF}^{ad})$.

Goal: $\{$vbs on $X_{FF}\}/\simeq \;\leftrightarrow\; \{$isocrys. $/\bar{\mathbb{F}}_p\}/=$ as a set, not equiv of cats.

[Recall: $X_{FF} = \mathrm{Proj}(P)$, $P := \bigoplus_{n\geq 0} B^{\varphi=p^n}$)

First: $\mathrm{Pic}(X_{FF}) \simeq \mathbb{Z}$

Key property: P is graded factorial w/ irred. elts in deg 1 (& these vanish w/ simple zero at one pt.) $(\approx \mathbb{P}^1_{\mathbb{C}}\,!)$

For any $m \in \mathbb{Z}$, $\exists\, \mathcal{O}(m) \in \mathrm{Pic}(X_{FF})$, def. by

$$\mathcal{O}(m) := \bigoplus_{n\geq m} B^{\varphi=p^n}. \quad \Big(= \bigoplus_{n\geq 0} (B\otimes D(-m))^{\varphi=p^n}\Big)$$

Note that $B^{\varphi=p^n} \simeq (B \otimes_{\breve{\mathbb{Q}}_p} D(-n))^{\varphi\otimes\varphi_n = id}$ ⚠ (works for any $D(\lambda)$)

X_{FF} is Dedekind

Prop: $\mathrm{Div}(X_{FF}) \xrightarrow{\sigma} \mathrm{Pic}(X_{FF})$ factors through $\mathbb{Z}$

$$\mathrm{Div}(X_{FF}) \xrightarrow{\deg} \mathbb{Z} \to \mathrm{Pic}(X_{FF}), \quad n \mapsto \mathcal{O}(n)$$

Pf: Use key property to show that $x \in X_{FF}$, $\mathcal{O}(x) \simeq \mathcal{O}(1)$. □.

Prop: $B^{\varphi=p^n} \xrightarrow{\sim} H^0(X_{FF}, \mathcal{O}(n))$.

Pf: next time. □.

Cor: $\mathrm{Pic}(X_{FF}) \simeq \mathbb{Z}$.

Now, let's construct $\mathcal{O}(\lambda)$ for $\lambda \in \mathbb{Q}$ ($\lambda = d/r$, $r>0$). (Rmk: $\mathcal{O}(\lambda) \leftrightarrow D(-\lambda)$).

- For any $E/\mathbb{Q}_p$ fin. unr., have $X_{FF,E}$; key property also works for $X_{FF,E}$

$$\begin{array}{ll} X_{FF,E_r} \simeq X_{FF} \otimes_{\mathbb{Q}_p} E_r & \\ \pi_r \downarrow \quad \text{fin. ét. of deg} & \Downarrow \\ X_{FF} \qquad [E:\mathbb{Q}_p] & \mathrm{Pic}(X_{FF,E}) \simeq \mathbb{Z} \end{array}$$

Fact: $E/\mathbb{Q}_p$ unr. $\overset{\deg h}{}$ then $X_{FF,E} \overset{:= \mathrm{Proj}(\bigoplus_{n\geq 0} B^{\varphi^h = p^n})}{\simeq} X_{FF} \otimes_{\mathbb{Q}_p} E. \simeq \mathrm{Proj}(P \otimes_{\mathbb{Q}_p} E)$

Pf: It will follow from: $B^{\varphi = p^n} \otimes_{\mathbb{Q}_p} E \simeq B^{\varphi^h = p^{nh}}$ (pick $\langle \gamma \rangle = \mathrm{Gal}(E/\mathbb{Q}_p)$-action on RHS by γ acting via $p^{-n}\varphi$) $\rightsquigarrow (B^{\varphi^h = p^{nh}})^{\mathrm{Gal}(E/\mathbb{Q}_p)} \simeq B^{\varphi = p^n}$. □.

Let E_h be the unr. extn. of $\mathbb{Q}_p$ of deg h.

Def: $\mathcal{O}(\lambda) := \pi_{r,*}(\mathcal{O}_E(d))$ It is a rk r v.b.

Rmk: In particular, $\pi_r^* \mathcal{O}(\lambda) \simeq \pi_r^* \pi_{r,*}(\mathcal{O}_E(d)) \simeq \mathcal{O}_E(d)^{\oplus r}$ ($E_r \otimes_{\mathbb{Q}_p} E_r \simeq E_r^{\oplus r}$

Def: $Z \in \mathrm{Sch}$ is a "complete alg curve":

1) Z is conn. sep., Noeth & reg. of dim 1
2) $\mathrm{Pic}(Z)$ admits a deg morphism taking positive values on effective $\neq 0$ Weil divisors

Def: Fix $\mathcal{F}$ a vb on Z. Then

1) $\deg(\mathcal{F}) := \deg(\Lambda^{\mathrm{rk}(\mathcal{F})} \mathcal{F})$ 2) For $\mathcal{F} \neq 0$, the slope $\mu(\mathcal{F}) := \deg(\mathcal{F})/\mathrm{rk}(\mathcal{F})$

have: $\mu(\mathcal{F} \otimes \mathcal{G}) = \mu(\mathcal{F}) + \mu(\mathcal{G})$

Prop: Given $u: \mathcal{F} \to \mathcal{G}$ map of vbs s.t. u is generically an iso. then

$$\deg(\mathcal{F}) \leq \deg(\mathcal{G})$$

w/ "$=$" $\Leftrightarrow$ u is an iso.

Def: $\mathcal{F} \neq 0$ v.b.

i) $\mathcal{F}$ is semistab. if $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$. $\forall\, \mathcal{G} \subset \mathcal{F}$ non-zero subbdle.

ii) $\mathcal{F}$ is stab. if — " — $<$ — " — $\forall\, \mathcal{G} \subsetneq \mathcal{F}$ — " — .

e.g. 1) All line bdles are stab.

2) For isocrystals, the stab objs are the $D(\lambda)$
semi-stab $D(\lambda)^{\oplus r}$

Prop: Each vb. $\mathcal{F}$ on Z has a unique functorial "Harder–Narasimham" filn:

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_r = \mathcal{F}$$

s.t. $\mathcal{F}_i$ are vbs & $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semi-stab. s.t. $\mu(\mathcal{F}_r/\mathcal{F}_{r-1}) < \cdots < \mu(\mathcal{F}_1)$

Pf: For $\mathcal{F}_1$, pick max. rk & deg. □

Prop: For $\lambda \in \mathbb{Q}$, the subcat. $\mathcal{C}_\lambda^{sst}$ (vbs semistab. of slope λ) is ab. & of fin length, i.e., each obj has a fin filn w/ simple graded quotients.

Pf: Any proper sub-obj must have strictly smaller rk & deg. $\square$