Talk 4: Schematic construction of the FF curve II

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Recall:

- fix $C^\flat$ ~~pts~~ perf. field in char p, $\varpi \in \mathfrak{m}_{\mathcal{O}_{C^\flat}}$ pseudo-unif.
- B = completion of $A_{\inf}[1/p, 1/[\varpi]]$ w.r.t. $\|\cdot\|_\rho$, $\rho \in (0,1]$
- View B as fcns on $|Y| = \{\text{char. } 0 \text{ untilts}\}/\sim\; = \mathrm{Prim}^1 / A_{\inf}^\times$, where $\mathrm{Prim}^k :=$ dist. elts of deg k $\subset A_{\inf}$

$$= \{\xi = \sum_{n\geq 0} [a_n] p^n \text{ s.t. } a_0 \neq 0,\ k = \text{smallest int. s.t. } a_k \in \mathcal{O}_{C^\flat}^\times\}$$

Note:

$$\mathrm{Prim}^k \cdot \mathrm{Prim}^\ell \subset \mathrm{Prim}^{k+\ell}$$

since $W(\mathcal{O}_{C^\flat}) \to W(\mathcal{O}_{C^\flat}/\mathfrak{m})$, $\mathrm{Prim}^k \mapsto \sum_{n\geq k} [\bar{a}_n] p^n$

- Think of $|Y|$ as $\mathbb{D}\setminus\{0\}$, $\{0\} \leftrightarrow C^\flat$

$$\mathbb{C}[1/z] \leftrightarrow A_{\inf}[1/p] \quad \text{analogy on } \mathbb{D}\setminus\{0\}$$

- Take $y = (K, \iota) \in |Y|$, (for some $a_0 \in \mathfrak{m}_{\mathcal{O}_{C^\flat}} \neq 0$)

$$\theta_y : W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_K \quad \text{w/ } \ker\theta_y = ([a_0] - p)$$

$\rightsquigarrow$ θ_y extends to $A_{\inf}[1/p, 1/[\varpi]] \to K$

<u>Goal</u>: extend θ_y further to B. Notation: $f \in B$, $f(y) := \theta_y(f)$

- $|Y|$ has an ultrametric given by ~~⚠ See FF's original article~~

$$d(y_1, y_2) = \inf_{r\in[0,1]} \{\mathfrak{p}_{y_1} + \mathfrak{a}_r = \mathfrak{p}_{y_2} + \mathfrak{a}_r\} \text{ where}$$

$$\mathfrak{a}_r = \{\sum_{n\geq 0} [a_n] p^n \in A_{\inf} : |a_n| \leq r\}$$

- $\mathfrak{p}_y = \ker(\theta_y)$
- $\mathfrak{a}_r = \{\sum_{n\geq 0} [a_n] p^n \in A_{\inf} : |a_n| \leq r, \forall n\}$ $= \{f \in A_{\inf} : v_0(f) \leq r\}$

In particular:

- $d(y, 0) = |a_0|$. $\tau(y) := d(y, 0)$
- $|Y_\rho| := \{y \in Y,\ d(y,0) \geq \rho\}$ for $0 < \rho < 1$
- $|Y_I| := \{y \in Y : d(y,0) \in I\}$ for $I \subset [0,1]$ cpt.
- $B := \varprojlim B_I$. B_I should be viewed as fcns on Y_I.

<u>Fact</u>: $\theta_y : A_{\inf}[1/p, 1/[\varpi]] \to C$ is cts w.r.t. $\|\cdot\|_{\tau(y)} := \sup_n |a_n| \tau(y)^n$

- B is complete w.r.t. all norms $\|\cdot\|_\rho$ $0 < \rho < 1$ $\Rightarrow$ θ_y can be extended to B.

<u>N.B.</u> We skip all the proofs using Newton polygon arguments.

<u>Lem.</u> B_I is a PID.

<u>Pf skh</u>: $0 \neq f \in B_I$, $f \notin B_I^\times \overset{NP}{\Longrightarrow} \exists\, x \in \mathfrak{m}_{C^\flat}$ s.t. $p - [x] \mid f$ w/ $|x| \in I$

$\Rightarrow f = (p-[x])g$, # slope of g = # slope of $f - 1$ $\Rightarrow f = \prod (p - [x_i])$

UFD + non inv. elts gen. max ideal $\Rightarrow$ PID ⌋

any ^prime

Cor: $|Y| \overset{1-1}{\longleftrightarrow}$ cl. max ideals of B

$y \mapsto P_y B = (\xi_y)$

⌈Pf: B_I PID $\Rightarrow$ MaxSpec $B_I \overset{1-1}{\longleftrightarrow} |Y_I|$

taking limits $\rightsquigarrow$ {cl. max'l ideal of B_I} $\longleftrightarrow$ $|Y| = \varprojlim |Y_I|$. ⌋

Goal: Upgrade this to a divisor version

- $A_{inf}[1/p, 1/[\varpi]] =: B^b \subset B$

$$(B^b)^{\varphi = p^n} = \begin{cases} 0, & n \neq 0 \\ \mathbb{Q}_p, & n = 0 \end{cases}$$

Prop: We have

$$B^{\varphi = p^n} = \begin{cases} 0 & \text{if } n < 0 \\ \mathbb{Q}_p & \text{if } n = 0 \\ \text{sth} \neq 0 & \text{if } n > 0 \end{cases}$$

$\curvearrowleft \mathbb{Q}_p$ -v.s.

e.g. $B^{\varphi = p} = 1 + \mathfrak{m}_{\mathcal{O}_{C^b}}$

$\alpha \cdot (1+x) := (1+x)^\alpha$, $\forall \alpha \in \mathbb{Q}_p$

For B_I (works for all PID),

$$(\text{Ideals of } B_I, \cdot) \xrightarrow[\text{monoid}]{\sim} \mathrm{Div}^+(|Y_I|) := \{\textstyle\sum_{y \in |Y_I|} n_y [y], \text{ fin. sum}\}$$

$$f \longmapsto \mathrm{div}(f) = \sum_{y \in |Y_I|} \mathrm{ord}_y(f)[y]$$

Prop: (non-zero cl. ideals of B, $\cdot$) $\xrightarrow[\text{monoid}]{\sim}$ $\mathrm{Div}^+(|Y|) := \varprojlim_I \mathrm{Div}^+(|Y_I|)$

$:= \{\sum_{y \in |Y|} n_y [y]$ s.t. support inside $|Y_I|$ is fin$\}$

Underlying top. sp. of the FF curve $|Y|/\varphi^{\mathbb{Z}}$, $\varphi(K, \iota) = (K, \iota \circ \varphi)$

Last time, we saw φ extends to B, so $\quad n_y = n_{\varphi(y)}$

$$\mathrm{Div}^+(|Y|/\varphi^{\mathbb{Z}}) = \{D \in \mathrm{Div}^+(|Y|), \varphi^* D = D\}$$
$$\hookrightarrow \mathrm{Div}^+(|Y|)$$

If $f \in B^{\varphi = p^k} \setminus \{0\}$, $\varphi^* \mathrm{div}(f) = \mathrm{div}(\varphi(f)) = \mathrm{div}(p^k f) = \mathrm{div}(f)$

$\uparrow$ p is inv.

$\Rightarrow \mathrm{div}: \bigsqcup_{k\geq 0} (B\setminus\{0\})^{\varphi=p^k} \longrightarrow \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$.

Thm: The map $\mathrm{div}: \left(\bigsqcup_{k\geq 0} (B\setminus\{0\})^{\varphi=p^k}\right)/\mathbb{Q}_p^\times \longrightarrow \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$ is an iso. (of monoids), which preserves deg. where

$$\deg\Big(\sum_{y\in Y/\varphi^{\mathbb{Z}}} n_y[y]\Big) := \sum n_y$$

Pf: (only works for $\mathbb{Q}_p$)

1) Injectivity: suppose $\mathrm{div}(f)=\mathrm{div}(g)$ w/ $f\in B^{\varphi=p^k}\setminus\{0\}$, $g\in B^{\varphi=p^{k'}}\setminus\{0\}$ w/ $k\geq k'$ WLOG. $\Rightarrow g/f \in B^{\varphi=p^{k-k'}} \Rightarrow k=k'$ & $f=g\cdot u$ for $u\in B^{\varphi=1}\setminus\{0\} = \mathbb{Q}_p^\times$

2) deg preserving: NP argument.

3) Surjectivity: ISTS: $\forall y\in|Y|$, $\exists t_y\in B^{\varphi=p}$ s.t.

$$\mathrm{div}(t_y) = \sum_{n\in\mathbb{Z}} [\varphi^n(y)]$$

non-triv. p-th root

Let (K,ι) be the untilt corresp. to y, take $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \cdots) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_K = \mathcal{O}_{K^\flat}$ is $\mathcal{O}_{C^\flat}$. Consider

$$\xi_y = 1 + [\varepsilon^{1/p}] + \cdots + [\varepsilon^{(p-1)/p}] = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1}$$

- $\theta_y(\xi_y) = 1+\zeta_p+\zeta_p^2+\cdots+\cancel{\zeta_p^{p-1}} = 0$
- $\varepsilon - 1 \in \mathfrak{m}_{\mathcal{O}_{C^\flat}} \Rightarrow \varepsilon^{1/p}-1, \varepsilon^{2/p}-1, \cdots, \varepsilon^{(p-1)/p}-1 \in \mathfrak{m}_{\mathcal{O}_{C^\flat}}$ $\mathcal{O}_{C^\flat}^\times$
- $W(\mathcal{O}_{C^\flat}) \to W(\mathcal{O}_{C^\flat}/\mathfrak{m}_{\mathcal{O}_{C^\flat}})$, $\xi_y \mapsto p \Rightarrow \xi_y \in \mathrm{Prim}^1 \Rightarrow \ker(\theta_y) = (\xi_y)$ $\overset{\text{Prop}}{\Rightarrow} \mathrm{div}(\xi_y) = [y]$.

$\xi_y = \cancel{1+}\underbrace{[\varepsilon^{1/p}]-1+\cdots+[\varepsilon^{(p-1)/p}]-1}_{\in \mathfrak{m}_{\mathcal{O}_{C^\flat}}} + p$

Consider $\log([\varepsilon]) = \sum_{k>0} \frac{(-1)^{k+1}}{k}([\varepsilon]-1)^k \equiv [\varepsilon]-1 \ (([\varepsilon]-1)^2)$. Claim: $\mathrm{div}(\underbrace{\log([\varepsilon])}_{t_y\in B^{\varphi=p}\setminus\{0\}}) = \sum_{n\in\mathbb{Z}} \varphi^n(y)$.

$\theta([\varepsilon]-1) = \zeta_p - 1 \in K^\times \Rightarrow \mathrm{ord}_y([\varepsilon]-1) = \mathrm{ord}_y(\xi_y)$

$\deg(\log[\varepsilon]) = 1$ $\Rightarrow \mathrm{ord}_y([\varepsilon]) = \mathrm{ord}_y([\varepsilon]-1) = 1$

$\Rightarrow \mathrm{div}(\log[\varepsilon]) = \sum_{n\in\mathbb{Z}} \varphi^n(y)$. $\square$

Cor: $1+\mathfrak{m}_{\mathcal{O}_{C^\flat}} \xrightarrow{\log} B^{\varphi=p}$ is an iso. of $\mathbb{Q}_p$-v.s.

Pf: Surj: $f\in B^{\varphi=p}$, $\mathrm{div} f = \sum_{n\in\mathbb{Z}} [\varphi^n(y)]$ for some $y\in Y \Rightarrow \mathrm{div}(f) = \mathrm{div}(\log[x])$ for some $x\in 1+\mathfrak{m}_{\mathcal{O}_{C^\flat}} \Rightarrow f = \log[x]\cdot g$ for some $g\in\mathbb{Q}_p^\times = \log([x]^g)$. $\square$

$X^{FF} = \mathrm{Proj}(\bigoplus_{k\geq 0} B^{\varphi=p^k})$.

Prop: $\bigoplus_{k\geq 0} B^{\varphi=p^k}$ is a graded factorial ring w/ irred elts in $B^{\varphi=p}$.

Pf: $f\in B^{\varphi=p^k}\setminus\{0\}$. $\mathrm{div}(f)=\sum_{i=1}^{k} D_i$. $\deg D_i = 1$, $D_i=\mathrm{div}(f_i)$, $f_i\in B^{\varphi=p}$ $\Rightarrow$ $\mathrm{div}(f)=\mathrm{div}(\prod f_i)$ $\Rightarrow$ $f=(\prod f_i)\cdot g$ w/ $g\in\mathbb{Q}_p^\times$. $\square$

Cor: $\{\text{pts of } X^{FF}\} \overset{1-1}{\longleftrightarrow} (B\setminus\{0\})^{\varphi=p}/\mathbb{Q}_p^\times \xrightarrow[\mathrm{div}]{\sim} |Y|/\varphi^{\mathbb{Z}}$

$\Uparrow$ purely alg. geo. fact

$(f) \longleftarrow f$

Toy example: $\mathbb{P}^1 = \mathrm{Proj}\bigoplus \mathbb{C}[x,y]_{\deg k} \simeq \mathrm{Proj}\bigoplus \mathbb{C}[X]_{\deg\leq k}$

$f \longmapsto f(X,1)$

$\mathbb{P}^1(\mathbb{C}) \simeq \mathbb{C}[x]_{\deg\leq 1}/\mathbb{C}^\times \longleftrightarrow$ lines passing through 0

Cor: X^{FF} is a Dedekind scheme.

Pf: Cover X^{FF} by $\mathrm{Spec}\, B[1/f]^{\varphi=1}$ w/ $f\in B^{\varphi=p}$.

- Fact: $B[1/f]^{\varphi=1}$ is a PID & res. field at a pt. $y=(K,\iota)\in D_+(f)$ is K

$\Rightarrow$ $B[1/f]^{\varphi=1}$ is reg. 1, dim 1, Noetherian

$\underbrace{B[1/f]^{\varphi=1}}_{} =: B_e =: B_{crys}^{\varphi=1}$ cls. period ring. $\square$

Cor:

For $y\in|Y|\leftrightarrow(K,\iota)$, x its image in $|Y|/\varphi^{\mathbb{Z}}$,

$$\hat{\mathcal{O}}_{X^{FF},x} \simeq \mathbb{B}^+_{dR,y} := \varprojlim_n \left(W(\mathcal{O}_{C^\flat})/\xi_y^n\right)[1/p]$$

w/ res field K.

Pf: Take $t\in B^{\varphi=p}\setminus\{0\}$ w/ $\mathrm{div}[t]=[x]$. Take $f\in B^{\varphi=p}\setminus\{0\}$ s.t. $f\notin\mathbb{Q}_p\cdot t$ $\rightsquigarrow$ $\mathrm{Spec}\, B[1/f]^{\varphi=1}\ni x$.

$B \longrightarrow \hat{B}_{\xi_y} \overset{\text{Thm}}{=} B^+_{dR,y}$ induces $B[1/f]^{\varphi=1} \longrightarrow B^+_{dR,y}$

$\Uparrow$ PID: $f \longmapsto (B^+_{dR,y})^\times$, $t/f \longmapsto$ uniformizer

on res. field both of them iso. to K

$\Rightarrow B[1/f]^{\varphi=1}_{(t)} \simeq B^+_{dR,y}$. $\square$

Reln w/ cls. period rings

Fix untilt $y=(K,\iota)$, ξ $\rightsquigarrow$ period rings $B^+_{crys} := \mathbb{A}_{inf}\left[\frac{\xi^n}{n!} : n\geq 1\right]^{\wedge p\text{-adic}}[1/p]$

$:=$ completion of $A_{\mathrm{inf}}[1/p]$

Fact: $\bigcap_{n\geq 0} \varphi^n(B^+_{crys}) = B^+ \subset B$

Thm: $B_{crys}^{+,\varphi=p^k} = B^{+,\varphi=p^k} \overset{NP}{=\!=} B^{\varphi=p^k}$.

$\Rightarrow X^{FF} = \mathrm{Proj}\left(\bigoplus_{k\geq 0} B_{crys}^{+,\varphi=p^k}\right)$. Now, consider $t \in B^{\varphi=p} \setminus \{0\}$, $\mathrm{div}(t) = [y]$
$B_{crys} = B^+_{crys}[1/t]$. $\Rightarrow B_e := B_{crys}^{\varphi=1} = B[1/t]^{\varphi=1}$, y $\hat{\mathcal{O}}_{X_{FF},y} \simeq B^+_{dR,y}$
$X_{FF} \setminus \{y\} = \mathrm{Spec}\, B_e$.