Talk 3

Recap: From perf. field K, tilting $\rightsquigarrow$ char p. perf. $K^\flat$

$\mathcal{O}_K \longmapsto \varprojlim \mathcal{O}_K/p =: \mathcal{O}_{K^\flat}$

Fix char p perf. $C^\flat$, goal: classify untilts (K, ι)

$$\begin{tikzcd} \mathbb{A}_{\inf} \arrow[r, "\theta"] \arrow[d, "{[\cdot]}"'] & \mathcal{O}_K \arrow[d] \\ \mathcal{O}_{C^\flat} \arrow[ur, dashed, "\#"] \arrow[r, "\#"'] & \mathcal{O}_K/p \end{tikzcd}$$

$\mathbb{A}_{\inf} = W(\mathcal{O}_{C^\flat})$

Any $x \in \mathbb{A}_{\inf}$ has a unique expansion $x = \sum_{i\geq 0}^{\infty} [c_i] p^i$ w/ $c_i \in \mathcal{O}_{C^\flat}$

$\theta(\sum [c_i] p^i) = \sum c_i^\# p^i$

- Think of $\mathbb{A}_{\inf}$ as a collection of fcns on the space of untilts (K, ι), i.e. evaluating $\sum [c_i] p^i$ at (K, ι) gives $\sum c_i^\# p^i$
- $\{$untilts of $C^\flat\}$/ isom $\longleftrightarrow$ $\{$distinguished ~~elt~~ elts $\xi \in \mathbb{A}_{\inf}\}/\mathbb{A}_{\inf}^\times$

$\mathrm{Frac}(\mathbb{A}_{\inf}/(\xi)) \longleftarrow\!\shortmid \xi$

$\xi = \sum [c_i] p^i$ w/ $|c_0|_{C^\flat} < 1$, $|c_1|_{C^\flat} = 1$

Write $\varpi = c_0$ quasi-uniformizer for $C^\flat$, then $\mathcal{O}_{C^\flat}/\varpi \simeq \mathcal{O}_K/p$; $|\varpi|_{C^\flat} = |p|_K \in (0,1)$

- Think of $|p|_K =: r(K, \iota)$ ~~as sth~~ measuring size of ϖ s.t. as organising untilts

$\rightsquigarrow$

Space of untilts	$\mathbb{D}$
$\lvert p\rvert_K$	$\lvert z\rvert$
$C^\flat$	0
$\mathbb{A}_{\inf}$	$\sum_{n\geq 0} c_n z^n$ s.t. $\lvert c_n\rvert \leq 1$
$\mathbb{A}_{\inf}[\frac{1}{[\varpi]}]$	$\sum_{n\geq 0} c_n z^n$ s.t. $\lvert c_n\rvert$ bdd
$\mathbb{A}_{\inf}[\frac{1}{[\varpi]}, \frac{1}{p}]$	$\sum_{n\geq -\infty} c_n z^n$ s.t. $\lvert c_n\rvert$ bdd
B	holo. fcn. on $\mathbb{D}^\times$

Enlarge $\mathbb{A}_{\inf}$

1) $\mathbb{A}_{\inf}[\frac{1}{[\varpi]}] \xrightarrow{\theta} \mathcal{O}_K[\frac{1}{p}] = K$ $\longleftarrow$ ⚠ indep. of choice of quasi-unif. ϖ since $|\cdot|_K$ is not discrete

In $\mathbb{A}_{\inf}[\frac{1}{[\varpi]}]$, have unique Teichmüller expansion

$\sum [c_n] p^n$ w/ $c_n \in C^\flat$ bdd.

① $= \sum \frac{[c_n \varpi^M] p^n}{[\varpi^M]}$

2) $\mathbb{A}_{\inf}[\frac{1}{[\varpi]}, \frac{1}{p}]$

Exp. $\sum_{n \gg -\infty} [c_n] p^n$

We'll define a further enlargement B: "ring of holo. fcns in var p"

Motivation:

① log. example

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

If $x \in C^b$, $1 > |x|_{C^b} = |x^\#|_K$, then $\log(1+x^\#)$ is well-def. in K.

Although

$$\mathbb{A}_{\inf}[\tfrac{1}{[\varpi]}] \xrightarrow{\theta} K$$

so in particular $\log(1+x^\#) = \theta(\text{smth})$, there is $\underline{\underline{\text{no}}}$ fcn. "$\log(1+[x])$" on $\mathbb{A}_{\inf}[\frac{1}{[\varpi]}, \frac{1}{p}]$ s.t. "evaluation" at (K, ι) gives $\log(1+x^\#)$.

② Complex analysis

- $\sum_{n \geq 0} [c_n] z^n$, $|c_n|$ bdd are holo. on $\mathbb{D}$, but can relax to $\limsup |c_n|^{1/n} \leq 1$
- $\sum_{n \gg -\infty} c_n z^n$, $|c_n|$ bdd are mero. at $0 \in \mathbb{D}$, but we want to allow holo. on $\mathbb{D}^\times = \mathbb{D} \setminus \{0\}$, i.e., we allow essential singularities at 0

$\updownarrow$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \text{ s.t. } \limsup_{\substack{n>0 \\ n \to \infty}} |a_n^{1/n}| \leq 1, \quad \lim_{\substack{n>0 \\ n \to \infty}} |a_{-n}^{1/n}| = 0$$

Def: Gauss norm $|\cdot|_\rho : \mathbb{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}] \to \mathbb{R}_{\geq 0}$, $(\rho \in (0,1))$ $\sum_{n \gg -\infty} [c_n] p^n \mapsto \sup\{|c_n|_{C^b} \rho^n\}$

If $c \in [a,b] \subset (0,1)$, then

$$|f|_c \leq \sup\{|f|_a, |f|_b\} \quad (\star)$$

$$\text{($n<0$ terms for } |f|_a\text{, } n>0 \text{ terms for } |f|_b\text{)}$$

Consider now

$$\sum_{n \in \mathbb{Z}} [c_n] p^n, \quad c_n \in C^b$$

this conv.

- wrt $|\cdot|_\rho \Leftrightarrow \lim_{n \to +\infty} |c_n|_{C^b} \rho^n \;\&\; \lim_{n \to +\infty} |c_{-n}|_{C^b} \rho^{-n} = 0$
- wrt all $|\cdot|_\rho \Leftrightarrow \limsup_{\substack{n>0 \\ n \to \infty}} |c_n|_{C^b}^{1/n} \leq 1 \;\&\; \lim_{n \to \infty} |c_{-n}|_{C^b}^{1/n} = 0$.
- wrt $|\cdot|_\rho$, $\rho \in [a,b] \Leftrightarrow$ conv. for $|\cdot|_a$ & $|\cdot|_b$

Def: B is the completion of $A_{inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$ w.r.t. $|\cdot|_p$, $\forall p \in (0,1)$.

$\Leftrightarrow$ $B = \varprojlim_{(a,b)} B_{[a,b]}$ w/ $B_{[a,b]} :=$ completion, — " — $|\cdot|_p$, $p \in [a,b]$

⚠

Q: What do elts of B look like?

If $c_n \in C^b$ w/ $\limsup_{\substack{n>0 \\ n\to\infty}} |c_n|_{C^b}^{1/n} \le 1$ & $\lim_{n\to+\infty} |c_{-n}|_{C^b}^{1/n} = 0$, then

$\sum_{n\in\mathbb{Z}} [c_n] p^n \in B$

But ! It's not clear if every elt of B has such a (unique) expn.

• Explicit description of $B_{[a,b]}$

Prop Suppose $\exists\, \varpi_a, \varpi_b \in C^b$ s.t. $|\varpi_a| = a$, $|\varpi_b| = b$, then

$$A_{inf}\left[\frac{[\varpi_a]}{p}, \frac{p}{[\varpi_b]}\right] = \left\{ f \in A_{inf}\left[\frac{1}{p}, \frac{1}{[\varpi]}\right] \text{ s.t. } |f|_a, |f|_b \le 1 \right\}$$

$$\left(\subseteq A_{inf}\left[\frac{1}{p}, \frac{1}{[\varpi]}\right] \right)$$

Cor: $B_{[a,b]} = \widehat{A_{inf}\left[\frac{[\varpi_a]}{p}, \frac{p}{[\varpi_b]}\right]} = \varprojlim A_{inf}[\cdots] / p^n A_{inf}[\cdots]$

• Frobenius: $\varphi: C^b \to C^b$, $c \mapsto c^p$

Goal: Lift to B

- $\varphi \in \mathrm{Aut}(\mathcal{O}_{C^b})$
- $\varphi \in \mathrm{Aut}(A_{inf})$ univ. property of $W(\mathcal{O}_{C^b})$
- $\varphi \in \mathrm{Aut}(A_{inf}[\frac{1}{p}, \frac{1}{[\varpi]}])$

Check: $|\varphi(f)|_{p^p} = |f|_p^{\,p}$, i.e. $\sup\{|c_n|^p p^{np}\} = \sup\{|c_n| p^n\}^p$

$\Rightarrow$ φ induces an iso. $B_{[a,b]} \simeq B_{[a^p, b^p]}$

$\Rightarrow$ $\varphi \in \mathrm{Aut}(B)$ w/ $\varphi(\sum [c_n] z^n) = \varphi([c_n^p] z^n)$

Consider $\bigoplus_{n\in\mathbb{Z}} B^{\varphi = p^n}$ w/ $B^{\varphi=p^n} := \{ f \in B : \varphi(f) = p^n f \}$.

e.g. 1) If $f \in B$ has exp $f = \sum_{n\in\mathbb{Z}} [c_n] p^n$ w/ limsup ≤ 1, lim $= 0$

then $f \in B^{\varphi = p^k}$ $\not\Rightarrow$ $\Leftarrow$

- $p^k f = \sum [c_n] p^{n+k} = \sum [c_{n-k}] p^n$
- $\varphi(f) = \sum [c_n^p] p^n$

$\Leftarrow$

$\not\Rightarrow$ $c_{n-k} = c_n^p$

! even if f has exp. the converse is not true bc. exp. is not unique

2) $k<0$ $\quad c_{n+mk} = c_n^{1/p^m}$ since $c_n \to 0$ as $n \to -\infty$, $c_n = 0$, $\forall n$

"no obvious elts of $B^{\varphi=p^k}$, $k<0$" In fact, $B^{\varphi=p^k} = 0$.

3) $k=0$. $c_n = c_n^p \Rightarrow c_n \in \mathbb{F}_p$ conv. $\Rightarrow c_n = 0$ for $n \ll 0$, i.e.,
$f \in W(\mathbb{F}_p)[\frac{1}{p}] = \mathbb{Q}_p \Rightarrow \mathbb{Q}_p \hookrightarrow B^{\varphi=p^0}$. In fact, $\mathbb{Q}_p \simeq B^{\varphi=p^0}$.

4) $k=1$ take $c \in \mathfrak{m}_{C^\flat}$, $\sum_{n\in\mathbb{Z}} [c^{1/p^n}] p^n \in B^{\varphi=p}$

Rmk: In $\mathbb{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$, $c_{n-k} = c_n^p \Rightarrow c_n = 0 \ \forall n$ (except $k=0$) since we have unique expr $\sum_{n \gg -\infty} [c_n] p^n$. So, only elts of $\mathbb{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}] \in B \oplus B^{\varphi=p^k}$ are $\mathbb{Q}_p$.

Def: The Fargues-Fontaine curve is $\mathrm{Proj}(\bigoplus_{n\geq 0} B^{\varphi=p^n})$.