Talk 2. Untilts, A_{inf} and B_{dR}^+

Last time: - perfectoid fields

- tilting: $K \mapsto K^\flat$ induces an equivalence of cats between $\{\text{fin. extns of } K\} \Leftrightarrow \{\text{fin. extns of } K^\flat\}$ preserving degs.

Q: Given a char p perf. field $C^\flat$, how do we find / characterize "untilts"?

Def: An untilt of $C^\flat$ is a pair (K, ι) where K is perf. & an iso. $\iota : K^\flat \xrightarrow{\sim} C^\flat$.

e.g. recall $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^\flat \simeq \widehat{\mathbb{F}_p((t^{1/p^\infty}))} \simeq \widehat{\mathbb{Q}_p(\zeta_p^{1/p^\infty})}^\flat$.

$$\widehat{\mathbb{F}_p((t^{1/p^\infty}))}^\flat$$

Rmk: An untilt is equivalent to $(K, \tilde{\iota})$, $\tilde{\iota} : \mathcal{O}_K/p \simeq \mathcal{O}_{C^\flat}/\varpi$ w/ $|\varpi| = |p|$. $|\varpi| < 1$

Motivation: Let (K, ι) be an untilt of $C^\flat$, let $x \in \mathcal{O}_K$. Then:

$$\begin{aligned} x &= c_0^\# + py \quad \text{w/ } c_0 \in \mathcal{O}_{C^\flat} \\ &= c_0^\# + pc_1^\# + p^2 y' \\ &\;\;\vdots \\ &= \sum_{i\geq 0} c_i^\# p^i \quad \text{highly non-unique} \end{aligned}$$

If K is alg. cl. then $x = c_0^\#$ is a choice of a system of compatible p-th power roots of x.

Witt vectors

- R a char p perfect ring (perfect $\mathbb{F}_p$-alg.)
- $W(R)$ is the unique ring s.t.
  1) $W(R)/pW(R) \simeq R$
  2) p is not a 0-divisor in $W(R)$
  3) $W(R)$ is p-adically complete

e.g. $W(\mathbb{F}_p) \simeq \mathbb{Z}_p$

Heuristically, $W(\mathbb{F}_p)$ is constructed w/ a clever choice of reps in $\mathbb{Z}_p/p \simeq \mathbb{F}_p$. Instead of $[0, \dots, p-1]$ roots of $x^{p-1}-1 = 0$ in $\mathbb{Z}_p$. This choice yields a multisection of $\mathbb{Z}_p/p \simeq \mathbb{F}_p$.

$$[\cdot] : \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$$

In general, we have the Teichmüller lift

$$[\cdot] : R \to W(R)$$

that satisfies: 1) $[x] \bmod p = x$ 2) $[xy] = [x][y]$

- $[x] = \lim_{n \to \infty} (\overline{x^{1/p^n}})^{p^n}$ where $\bar{\cdot}$ is some lift to $W(R)$.
- When $R = \varprojlim \mathcal{O}_K/p$, and we lifted to $\mathcal{O}_K$, this is the # map.
- We get a Teichmüller decomp.: $\underset{\underset{W(R)}{\uparrow}}{x} = \sum [\underset{\underset{R}{\uparrow}}{c_i}] p^i$ <u>UNIQUELY</u>.
- <u>Universal</u> property: For any p-adically complete ring A,

$$\mathrm{Hom}(W(R), A) \simeq \mathrm{Hom}(R, A/pA).$$

<u>Def</u>: $A_{\inf} := W(\mathcal{O}_{C^\flat})$

We get a map $\varphi : \mathbb{A}_{\inf} \to \mathbb{A}_{\inf}$

$$\sum [c_i] p^i \mapsto \sum [c_i^p] p^i$$

Fix an untilt

$\# : \mathcal{O}_{C^\flat} \to \mathcal{O}_K$ is not a morphism of rings, but it is mod ϖ. By univ. property, $\exists$:

$$\theta : \mathbb{A}_{\inf} \twoheadrightarrow \mathcal{O}_K \qquad \mathcal{O}_{C^\flat} \xrightarrow{\sim} \mathcal{O}_{K^\flat} \xrightarrow{\#} \mathcal{O}_K$$

$$\sum [c_i] p^i \mapsto \sum c_i^\# p^i \qquad \theta \text{ is local}$$

$\rightsquigarrow$

$$\begin{tikzcd} \mathbb{A}_{\inf} \arrow[r, two heads, "\theta"] \arrow[d, "\bmod p"'] & \mathcal{O}_K \arrow[d, "\bmod p"] \\ \mathcal{O}_{C^\flat} \arrow[r, "\#"'] \arrow[u, bend left, dashed, "{[\cdot]}"] \arrow[ur, "\#"] & \mathcal{O}_K/p \end{tikzcd}$$

(# induces $\mathcal{O}_{C^\flat}/\varpi \simeq \mathcal{O}_K/p$)

<u>e.g.</u> Fix $|\underset{\underset{\mathcal{O}_{C^\flat}}{\uparrow}}{\varpi}| = |p|$ $(= p^\flat)$ so $\varpi^\# = p\underset{\underset{\mathcal{O}_K^\times}{\uparrow}}{\bar{u}}$ let $u \in \mathbb{A}_{\inf}$ be s.t. $\theta(u) = \bar{u}$. Then $[\varpi] - up$ is in $\mathrm{Ker}(\theta)$.

<u>Def</u>: An elt $\xi \in \mathbb{A}_{\inf}$ is <u>distinguished</u> if $\xi = [\varpi] - up$, w/ $u \in \mathbb{A}_{\inf}^\times$ and $|\varpi| < 1$.

i.e. $\xi = \sum_{i \geq 0} [c_i] p^i$, $|c_0| < 1$, $|c_1| = 1$.

<u>Prop</u>: Let $C^\flat$ be a perf. field in char p, $\xi \in \mathbb{A}_{\inf}$ distinguished elt.

Then $A_{\inf}/(\xi) \simeq \mathcal{O}_K$ for a perf. field K. Moreover,

$$\mathcal{O}_{C^\flat} \simeq A_{\inf}/(p) \to A_{\inf}/(\xi, p) \simeq \mathcal{O}_K/(p)$$

realizes K as an untilt of $C^\flat$.

Cor: Let K be an untilt. Then $\ker(\theta)$ is principal & gen by ξ.

Pf: $A_{\inf}/(\xi) \xrightarrow{\theta} \mathcal{O}_K$

is $\mathcal{O}_{K'}$ $\Rightarrow \ker(\bar\theta)$ is prime $\subset \mathcal{O}_{K'}$ — $\times$ bc. $\mathcal{O}_K$ is not a field.

$\Rightarrow \ker(\bar\theta)$ is either 0 or maximal

$\rightsquigarrow$ $\{$dist. elts in $A_{\inf}\}$ / mult. by unit $\overset{\sim}{\longleftrightarrow}$ $\{$untilts of $C^\flat\}$ / iso.

Proof of Prop: 1) Prove that $A_{\inf}$ is ξ-torsion free, ξ-adically free & $A_{\inf}/\xi$ is p-torsion free, p-adically complete.

$\quad$ follows from the same w/ $p \rightleftarrows \xi$

2) Prove that $\forall y \in A_{\inf}/\xi$, $\exists x \in \mathcal{O}_{C^\flat}$ s.t. $y = x^\# u$ w/ $x^\# = [x] \bmod$

3) Prove that this x is "not too far from unique." & multn by a unit is the or freedom.

$$x, x' \in \mathcal{O}_{C^\flat} \text{ s.t. } x^\# \mid x'^\# \Rightarrow x \mid x' \quad (|x| \geq |x'|)$$

4) $A_{\inf}/(\xi)$ is integral

$\Rightarrow$ Can define $|\cdot| : A_{\inf}/(\xi) \to \mathbb{R}$, $y \mapsto |y| := |x|_{C^\flat}$ (well-def. by 3

Then check that $|\cdot|$ is an abs. val. &

$$|y| \leq |z| \Leftrightarrow z \mid y$$

$\Rightarrow$ strong triangle inequality.

$\Rightarrow (K, |\cdot|)$ non-Arch. field

$\quad \overset{..}{}$ $\mathrm{Frac}(\underbrace{A_{\inf}/\xi}_{::\ \mathcal{O}_K})$

- $|p| = |\varpi|_{C^\flat} < 1 \Rightarrow$ K has res char p. & K has char 0
- $\mathcal{O}_K$ is p-adically complete $\Rightarrow$ K is complete
- $m \neq (p)$ $(\varpi^{1/p})^\#$ is not a multiple of p
- $\mathcal{O}_K/(p) \simeq \mathcal{O}_{C^\flat}/(\varpi)$

$\quad$ is $A_{\inf}/(\xi, p)$ $\quad$ is

We will define a normalization of $|\cdot|_K$ by its compatibility w/ $\mathcal{O}_{C^\flat} \Rightarrow$

tells you sth about K. $\rightsquigarrow$ Identify "$Y = \{\text{untilts}\}/\text{iso} \leftrightarrow \mathbb{D} = \{|z|<1\}$" $\quad (K, v) \leftrightarrow |p|_{\varpi}$

$\sum [c_i]p^i = \dfrac{\sum [c_i \varpi^n]p^i}{[\varpi^n]}$ $\leftarrow$ $\left\{ \sum [c_i]p^i,\ c_i \in C^{\flat} \right\}$ $\quad |c_i|$ bdd

$A_{inf} \xrightarrow{\theta} O_K$ extends to $A_{inf}[1/[\varpi]] \to O_K[1/\varpi^{\#}] = K$. If we exclude the triv. untilt, this factors through

$$A_{inf}[1/[\pi], 1/p] \longrightarrow K$$

$$\Vert$$

$$\left\{ \sum_{i >> -\infty} [c_i]p^i,\ |c_i| \text{ bdd} \right\}$$

- $B_{[a;b]}$ fcns on the strip $[a;b]$
- Take some limit to get B "holo. fcns in var. p"
- To any untilt $\hat{y}$ of y char 0, assoc. to it the $\widehat{\text{fcns}}$ of B def at y "$\hat{\mathcal{O}}_{\mathbb{D},y} = B^{+}_{dR}(y)$"