Talk 1.

Def: Let K be a non-arch. field of res. char p w/ non-discrete valn. $\mathcal{O}_K$ ring of ints. K is perfectoid iff it is complete & $\phi : \mathcal{O}_K/p \to \mathcal{O}_K/p$, $x \mapsto x^p$ is surj.

e.g. 1) $\mathbb{Q}_p$, $\mathbb{F}_p((t))$ ✗

2) $\mathbb{C}_p$ ✓

3) $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ ✓ $\mathcal{O}_K = \widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$
$\quad \| \quad$
$\quad K$

4) $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ ✓ $\mathcal{O}_K = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$

5) $\widehat{\mathbb{F}_p((t))(t^{1/p^\infty})}$ ✓ $\mathcal{O}_K = \widehat{\mathbb{F}_p[[t^{1/p^\infty}]]}$

Tilting

- K perfectoid, fix $\varpi \in \mathcal{O}_K$ w/ $|p| \le |\varpi| < 1$
- Consider $\varprojlim_{\phi} \mathcal{O}_K/\varpi$ ← well-def. ring str.

$$\begin{array}{c} \Psi \\ x = (\bar{x}_0, \bar{x}_1, \cdots) \end{array}$$

where $x_i \in \mathcal{O}_K$ & $\bar{x}_i^{\,p} = \bar{x}_{i-1}$. Any x has a unique p-th root $x^{1/p} = (\bar{x}_1, \bar{x}_2, \cdots) \Rightarrow \varprojlim_{\phi} \mathcal{O}_K/\varpi$ is perfect. + limit of discrete top.

Prop: $\exists$ a cts multiplicative map

$$\varprojlim_{\phi} \mathcal{O}_K/\varpi \xrightarrow{\#} \mathcal{O}_K$$

Pf: Let $x = (\bar{x}_0, \bar{x}_1, \cdots)$. Choose lifts x_i for $\bar{x}_i$. Claim:

$$\lim_{n\to\infty} x_n^{p^n} \text{ is well-def. \&}$$

does not depend on the choice of x_i.

Let x_n' be another lift of $\bar{x}_n$, then $x_n^{p^i} - x_n'^{p^i}$ is div. by ϖ^{i+1}, $\forall i \in [0, n]$.

Induction: ① ✓ for $i=0$.

② $x_n^{p^i} - x_n'^{p^i} = (x_n'^{p^{i-1}} + \underbrace{(x_n^{p^{i-1}} - x_n'^{p^{i-1}})}_{\varpi^i y})^p - x_n'^{p^i}$

$$= p\varpi^i y ((x_n')^{p^{i-1}})^{p-1} + \lambda \varpi^{2i} y + \cdots$$

$$= \varpi^{i+1} y' + \cdots \quad ✓$$

Now, $x_n^{p^n} - x_{n-1}^{p^{n-1}}$ is div. by ϖ^n

note that $\hat{x_n^p}$ is another lift of $\bar{x}_{n-1}$ $\square$.

Write $x^\# = \lim_{n\to\infty} x_n^{p^n}$.

Redn mod ϖ

Cor: $\vee$ is a multiplicative hom.

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_K \xrightarrow{\sim} \varprojlim_{\phi} \mathcal{O}_K/\varpi$$

w/ inverse given by $\#$: $x \in \varprojlim_{\phi} \mathcal{O}_K/\varpi$, we associate to the elt. $(x^\#, (x^{1/p})^\#, \cdots)$

Rmk: 1) $\#$ is multiplicative but <u>not</u> additive.

$$(x+y)^\# = \lim_{n\to\infty} ((x^{1/p^n})^\# + (y^{1/p^n})^\#)^{p^n} \quad (\star)$$

It is additive mod ϖ.

2) $\#$ is <u>not</u> surj. Anything in its image has a system of p-power th. roots.

3) $(\star)$ defines addition on $\varprojlim_{x\to x^p} \mathcal{O}_K$.

$$(x+y)_n := \lim_{m\to\infty} (x_{n+m} + y_{n+m})^{p^m}.$$

Prop: $\exists$ an elt. $\varpi^b \in \varprojlim_{\phi} \mathcal{O}_K/\varpi$ s.t. $|(\varpi^b)^\#| = |\varpi|$.

Pf: Take $\varpi_1 \in \mathcal{O}_K$ s.t. $|\varpi_1^p| = |\varpi|$. Suppose $|\varpi| > |p|$, then by surj of ϕ on $\mathcal{O}_K/p$, we get $\varpi = y^p + pz$. Take $\varpi_1 = y$. If $|\varpi| = |p|$, then $\exists\ x$ s.t. $|p| < |x| < 1$. Write $\varpi = xy$ w/ $|p| < |y| < 1$. $\lrcorner$

Consider any $\underbrace{(0, \bar{\varpi}_1, \cdots)}_{\varpi^b} \in \varprojlim \mathcal{O}_K/\varpi$.

$$|(\varpi^b)^\# - \varpi_1^p| \le |\varpi|^2.$$

Note that $((\varpi^b)^{1/p})^\#$ is another lift of $\bar{\varpi}_1$. $\Rightarrow |(\varpi^b)^\#| = |\varpi|$. $\square$.

We now fix $\varpi = (\varpi^b)^\#$. It has ϖ^{1/p^n} well-def.

$\underline{\underline{\text{Def}}}$: $K^b = \underbrace{\varprojlim_{x \mapsto x^p} \mathcal{O}_K}_{\mathcal{O}_K^b} [(\varpi^b)^{-1}]$ (note $K = \mathcal{O}_K[\varpi^{-1}]$)

$\#$ extends to a map $K^b \to K$.

$\underline{\text{Thm}}$ 1) K^b is a perfect field of char. p. $K^b \simeq \varprojlim_{x \to x^p} K$

2) $|\cdot|_{K^b} : K^b \to |K|$ is an a.v. & makes K^b perfectoid.

$x \mapsto |x^\#|_K$

3) Moreover, $|K^b| = |K|$, $\mathcal{O}_{K^b} = \mathcal{O}_K^b$; $\mathcal{O}_{K^b}/\varpi^b \overset{u_0}{\simeq} \mathcal{O}_K/\varpi$, and $\mathcal{O}_{K^b}/m \simeq \mathcal{O}_K/m$.

$\underline{\underline{\text{Pf}}}$: ① triangular inequality. use formula for addition.

② $\underline{\text{completeness}}$: consider $\mathcal{O}_{K^b} \to \mathcal{O}_K$, $x \mapsto (x^{1/p^m})^\#$

$$\begin{array}{c} \downarrow \\ \mathcal{O}_K/\varpi \end{array} \qquad y \mapsto y \bmod \varpi$$

This factors through $\mathcal{O}_{K^b}/(\varpi^b)^{p^m}$ See e.g. Lem 3.2.2 of Bhatt's notes Cor 3.2.3

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{O}_{K^b}/(\varpi^b)^{p^m} \arrow[r, "\phi"] \arrow[d, "u_m"] & \mathcal{O}_{K^b}/(\varpi^b)^{p^{m-1}} \arrow[r] \arrow[d, "u_{m-1}"] & \cdots \\ \cdots \arrow[r] & \mathcal{O}_K/\varpi \arrow[r, "\phi"] & \mathcal{O}_K/\varpi \arrow[r] & \cdots \end{tikzcd}$$

The limit of the top row being $\mathcal{O}_{K^b} \Leftrightarrow$ completeness

$\underline{\underline{Ex}}$: $K = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$, $\mathcal{O}_K = \widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$

- $\mathcal{O}_K/p = \mathbb{Z}_p[\zeta_{p^\infty}]/p$

$= \mathbb{Z}_p[x_1, x_2, \cdots]/(x_1^{p-1} + x_1^{p-2} + \cdots + 1,\ x_2^p = x_1,\ x_3^p = x_2, \cdots)/p$

$= \mathbb{F}_p[x, x^{1/p}, x^{1/p^2}, \cdots]/(x^{p-1} + \cdots + x + 1)$

$= \mathbb{F}_p[x^{1/p^\infty}]/(\frac{x^p - 1}{x-1})$

$= \mathbb{F}_p[x^{1/p^\infty}]/(x-1)^{p-1}$

$= \mathbb{F}_p[(1+t)^{1/p^\infty}]/t^{p-1} = \mathbb{F}_p[t^{1/p^\infty}]/t^{p-1}$

- $\mathcal{O}_{K^\flat} = \varprojlim_{\phi} \mathcal{O}_K/p = \varprojlim_{\phi} \mathbb{F}_p[t^{1/p^\infty}]/t^{p-1} = \widehat{\mathbb{F}_p[[t^{1/p^\infty}]]}$

$\Rightarrow K^\flat = \widehat{\mathbb{F}_p((t^{1/p^\infty}))}$ where t corresp. to $(1, \zeta_p, \zeta_{p^2}, \cdots) - 1 \in \varprojlim_{x \mapsto x^p} K$

$\underline{\underline{Ex}}$: $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$, $\mathcal{O}_K = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$

- $\mathcal{O}_K/p = \mathbb{F}_p[t^{1/p^\infty}]/t$

$\Rightarrow K^\flat = \widehat{\mathbb{F}_p((t^{1/p^\infty}))}$

$\underline{Rmk}$: 1) $K = \mathbb{C}_p$, $K^\flat = \widehat{\overline{\mathbb{F}_p((t))}}$. $\underline{Thm}$: Tilting yields an equivalence of cats. between extns of K & extns of $K^\flat$ $\Rightarrow$ alg. cl. $\xleftrightarrow{\flat}$ alg. cl.

2) $\underline{\text{Tilting equiv.}}$: One can play the same game w/ R, R° instead of K, $\mathcal{O}_K$ & obtain $R^\flat$, $R^{\flat\circ} = R^{\circ\flat}$ over $K^\flat$

R: K-alg, R°: power bdd. elts

$\underline{Thm}$: Tilting induces an equiv. of cats from K-PerfAlg to $K^\flat$-PerfAlg