# Study Group on Eigenvarieties

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**Introduction.** The concept of an eigenvariety lies at the heart of the modern approach to $p$-adic families of automorphic forms. Roughly speaking, an eigenvariety is a rigid-analytic or adic space that parametrizes systems of Hecke eigenvalues appearing in spaces of overconvergent automorphic forms, as these vary over a family of $p$-adic weights.

The story begins with the discovery that modular forms, traditionally defined over the complex numbers, can be $p$-adically interpolated. This idea was pioneered by Serre in the 1970's in his beautiful paper [Ser73], where he introduced the first definition of a $p$-adic (family of) modular form(s). In particular, Serre showed that a single such family – the $p$-adic family of Eisenstein series – gives rise to numerous congruences between modular forms and yields an immediate construction of the Leopoldt–Kobuta $p$-adic zeta function. Almost around the same time, Katz gave a geometric construction via the theory of formal schemes [Kat73], providing a generalization of Serre's approach.

However, it took more than a decade before further applications of $p$-adic families were realized, largely because it was not clear how to construct $p$-adic families of modular eigenforms beyond the "ur-example" of the Eisenstein family. A major breakthrough came with the work of Hida, who provided a rich theory of $p$-adic families of modular forms in the *ordinary* case (e.g. [Hid86]). Though limited to the ordinary setting, this results had far-reaching applications such as Greenberg–Steven's proof of the Mazur–Tate–Teitelbaum conjecture for elliptic curves. Hida also studied $p$-adic analytic families of Galois representations attached to ordinary eigenforms. His ideas inspired the work of Mazur and Wiles on families of Galois representations, and consequently, Mazur's theory of deformation of Galois representations which turned out to be a crucial ingredient of Wiles's proof of Fermat's Last Theorem.

Another decade passed before the "ordinary" restriction which appeared in Hida's work could be eased in any substantial way. In the 1990s, Coleman developed a rigid-analytic theory of overconvergent modular forms, which was essentially built on the earlier work of Katz [Kat73]. Coleman [Col97], [Col96] proved the existence of many families: almost every overconvergent eigenform of finite slope lives in a $p$-adic family. Coleman's important result [Col96] that overconvergent modular forms of small slope are classical, enabled him to prove the existence of an abundance of $p$-adic families of classical modular forms. These results were motivated by, and gave answers to, a variety of questions and conjectures that Gouvêa and Mazur had made based on ample numerical evidence.

This theory reached an aesthetic culmination in Coleman–Mazur's [CM98] construction of the *eigencurve*, a rigid-analytic curve whose points correspond to finite-slope $p$-adic overconvergent eigenforms of a fixed tame level $N$.

Since their seminar work, eigenvarieties have attracted broad interest over the past two decades. Buzzard [Buz07] axiomatized Coleman–Mazur and proposed the so-called *eigenvariety machine*, flexible framework that takes as input a coherent system of Hecke actions on Banach modules varying over weight space, and outputs a geometric space interpolating the systems of Hecke eigenvalues. Several approaches instantiate this framework in different settings:

◦ The Coleman–Mazur's original construction, via the geometric theory of modular forms, has been extended to the contexts of Shimura curves by Kassaei [Kas04], to Hilbert modular varieties by Kisin and Lai [KL05], and to Siegel modular varieties by Andreatta–Iovita–Pilloni [AIP15], just no name a few.

◦ Pollack–Stevens [PS99] ($\mathrm{GL}_2/\mathbb{Q}$) and Ash–Stevens [AS08] (connected reductive group split at $p$) constructed eigenvarieties using overconvergent cohomology theories. These constructions built on Stevens' beautiful idea of modular symbols (a group-cohomological avatar of overconvergent $p$-adic modular forms), and have been generalized to broader classes of reductive groups by the works of Urban [Urb11] Hansen [Han14], and others.

◦ Emerton [Eme06] introduced a third perspective (related to the second), constructing eigenvarieties from his completed cohomology and the interpolation of locally analytic vectors.

Eigenvarieties now serve as a powerful bridge between geometry, representation theory, and arithmetic. They have played an important role in the development of the $p$-adic Langlands program, the study of $p$-adic $L$-functions, and the arithmetic of Galois representations.

See the survey articles by Kassaei [Kas05], Emerton [Eme06], Vonk [Von21], as well as section 3 of Newton's notes [HLV24], and the works mentioned above, for furthuer details and a more thorough account of the literature.

**Goals.** The purpose of this study group is twofold: First, we aim to understand the construction, geometry, and some applications of eigenvarieties using the modern language of adic spaces. Building on this foundation, we then hope to explore further developments in this fascinating area of research, such as:

◦ Ash–Stevens' overconvergent modular symbols,

◦ Emerton's approach to eigenvarieties via completed cohomology,

◦ perfectoid techniques,

◦ · · ·

depending on the group's interest (but yeah, we are curious about pretty much everything). Promising materials include:

- The recent 2023 Heidelberg Spring School notes [HLV24] "Non-Archimedean Geometry and Eigenvarieties", available on arXiv and published by EMS last year. This will be our main reference for the first and (part of) second sections.

- Bellaïche's "Eigenbook" [Bel21], which offers a modern and systematic treatment of eigenvarieties and overconvergent modular symbols (and much more).

- Works of Buzzard [Buz07], Emerton [Eme06], Urban [Urb11], and Hansen [Han14], among others.

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Yicheng and I hope to organize follow-up a study group on higher Hida and Coleman theories developed by Boxer–Pilloni [BP20], [BP21], [BP22] in the future.

**Time & Location.** To be determined (7:30-9:30pm BJT, 1:30-3:30pm CET), online via Zoom.

**Tentative schedule.** The following is a rough outline and includes suggestions for each talk. The organizers are happy to discuss and adjust the materials or structure of the talks, especially if you find that too much or too little content is assigned in any single session. We are flexible with the schedule. *After all, we are organizing this study group so that we can enjoy learning some beautiful mathematics together—not to sprint ahead until no one has a clue of what's going on anymore.*

The interested participant may compare with the seminar programs on similar materials but from different perspectives, here and here.

**Talk 0 - Prologue: geometry of eigenvarieties.** (July 21, Haoran) Give a quasi-historical introduction to the theory of $p$-adic modular forms and some motivation behind eigenvarieties, following [Eme09] and [Cal13]. Outline and highlight the works of Serre [Ser73], Katz [Kat73], Hida [Hid86], [Hid93], Coleman [Col96], Coleman–Mazur [CM98], et. al. Topics:

- $p$-adic modular forms à la Serre
- Katz's geometric formulation
- overconvergent modular forms and classicality
- $p$-adic families of modular forms and the eigencurve
- a roadmap of the study group

Part I – Foundations and Machinery

**Talk 1 - Adic Spaces I: Basic definitions and examples.** (July 29, Lichang) Introduce the building blocks of adic spaces. Topics:

- From rigid analytic geometry to adic spaces – Tate, Raynaud, Berkovich, Huber
- Huber rings and valuations; the adic spectrum
- Rational subsets and the structure (pre-)sheaf
- The closed unit disc (a key example)

This talk could follow closely the first two chapters of the Heidelberg notes [HLV24] and aim to make these new spaces feel concrete. (It might be helpful to consult Conrad's notes [Con08] for Tate, Raynaud, and Berkovich's approaches.)

**Talk 2 - Adic Spaces II: geometry of adic spaces.** (August 04, Julia) Building up on the theory of Huber pairs from Talk 1, develop the general construction of adic spaces. Discuss some important classes of adic spaces such as rigid analytic spaces and formal schemes, and explore the connections between them. Illustrate the respective concepts with the fundamental examples of the open and closed disc and the affine line. Again, a helpful resource is [HLV24] (the first three chapters).

- Definition of general adic spaces

- ◦ Morphisms of adic spaces
- ◦ Comparison theorems – adic vs rigid vs formal vs Berkovich
- ◦ Basic sheaf theory and coherent sheaves

**Talk 3 - Digression: perfectoid geometry.** (Ho Leung) Following chapter 4 of [HLV24] ntroduce perfectoid fields, perfectoid algebras, and the tilting equivalence. Emphasize conceptual motivations and key phenomena.

- ◦ Motivations from the Fontaine–Wintenberger Theorem
- ◦ Definition and examples of perfectoid spaces
- ◦ *Applications to $p$-adic Hodge theory, the Langlands program, etc., if time permits

Possible further readings: Saito's survey article [Sai14], Scholze's and Weinstein's AWS 17 lectures, Bhatt's lecture notes on perfectoid spaces, ...

**Talk 4 - The eigenvariety machine I: toolkit from functional analysis.** (Zhenghang) Reference: [HLV24], sections 2 and 4 of chapter 5. [Buz07] also contains a very readable exposition in the rigid analytic language. Introduce the necessary functional analysis for the construction of eigenvarieties. Ccentral concepts are Banach–Tate rings, (orthonormalisable) modules over them, and the theory of compact operators on these modules. These provide the technical framework to handle the input data for the eigenvariety machine. This talk is geometry-free. Topics:

- ◦ Fredholm determinants and their properties
- ◦ Riesz theory
- ◦ Slope decompositions

**Talk 5 - The eigenvariety machine II: construction and first properties.** (Haoran) Introduce the weight space (see [HLV24], section 2.3 of chapter 6; while the rest can be found in chapter 5 loc. cit.). Construct spectral varieties over an affinoid base, which sits between the eigenvariety and the weight space. Discuss geometric properties of the structural map from the spectral variety to the weight space. Outline the construction of eigenvarieties and state their first properties. If time permits, discuss the case of $G = \mathrm{Res}_{L/\mathbb{Q}}(\mathrm{GL}_1)$ where $L$ is a number field (See [Buz04]).

- ◦ weight space
- ◦ spectral varieties over an affinoid base; the structural map
- ◦ construction of eigenvarieties and their first properties
- ◦ a toy example: $G = \mathrm{Res}_{L/\mathbb{Q}}(\mathrm{GL}_1)$

Part II: Key Examples and Techniques

**Talk 6 - The Coleman–Mazur eigencurve I: automorphic side.** (Haoran) Recall Coleman's theory of $p$-adic families of overconvergent modular forms and discuss Coleman's classicality results [Col96]. Present the construction of the Coleman–Mazur eigencurve prove some of its geometric properties, following section 2 of Newton's notes in [HLV24]. If time permits, survey some questions regarding the geometry of the eigencurve, such as components, the ghost conjecture (see [BG16], the work of Liu–Wan–Xiao [LWX17] and Liu–Truong–Xiao–Zhao [LTXZ23]), etc.

- ◦ $p$-adic and overconvergent modular forms
- ◦ classicality theorems
- ◦ definition and properties of the Coleman–Mazur eigencurve
- ◦ *global geometric features

**Talk 7 - The Coleman–Mazur eigencurve II: Galois side.** (Yicheng) ??? [TBD]

- ◦ pseudo-representations
- ◦ Galois deformations
- ◦ families of (pseudo-)representations on the eigencurve
- ◦ infinite fern

**Talk 8 - Eigencurves for definite quaternion algebras** (Benchao). Following section 4 of Newton's notes in [HLV24], discuss overconvergent automorphic forms for definite quaternion algebras (over $\mathbb{Q}$) and their associated eigencurves. This is in many ways simpler, since the Shimura curve involved is of dimension 0 (a finite connection of points).

- ◦ automorphic forms for definite quaternion algebra
- ◦ (families of) overconvergent automorphic forms
- ◦ eigencurves for definite quaternion algebras and comparison with the Colemann–Mazur eigencurve (Jacquet—Langlands)

Part III – Further developments

**Talks 9-11 (?) Variant I: Stevens' overconvergent modular symbols.** (Pengcheng, Zhenghang+) Following part II of [Bel21], explore an alternate construction of the eigencurve via overconvergent modular symbols and compare it with the Coleman–Mazur eigencurve. Topics:

- ◦ modular symbols - classical and $p$-adic
- ◦ distribution modules and analytic continuation
- ◦ construction of eigenvarieties via modular symbols
- ◦ comparison with the Coleman–Mazur eigencurve

**Talks 12-14 (?). Variant II: Emerton's construction via completed cohomology.** (Benchao, Deding, Yicheng) Following [Eme06], introduce Emerton's perspective on eigenvarieties, constructed from completed cohomology of modular curves. Discuss the interpolation of Hecke eigenvalues via locally analytic vectors and Jacquet modules. Explain how this gives rise to eigenvarieties compatible with the eigenvariety machine but arising from representation theory.

??? [TBD]

Bonus talk: If people are interested, Haoran is happy to give a talk on the local geometry of eigenvarieties on the work of Bellaïche–Dimitrov [BD16], its further developments and applications at the end of this study group.

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