We have $AJ: X_0(N) \longrightarrow J_0(N)$, construct A by quotienting $J_0(N)$

Need to check:

- $J_0(N)$ has good reduction outside N

b/c $X_0(N)$ has a smooth model over $\mathbb{Z}[\frac{1}{N}]$

- (Today) $J_0(N)$ has toric reduction at N.

This is sufficient because toric and good reductions pass through quotients:

Let A be an AV over $K = \mathrm{Frac}(R)$, R: DVR, k: residue field

If $f: A \longrightarrow B$ isogeny of AVs, $\exists\ g: B \longrightarrow A$ s.t. $fg = [n]$

$\mathcal{A}, \mathcal{B}$: Néron models of A and B

NMP $\Rightarrow$ f, g extends to $\mathcal{A}$ and $\mathcal{B}$ and $fg = [n]$ still holds

If A has semistable (good) reduction then $[n]: \mathcal{A}_k \longrightarrow \mathcal{A}_k$ is surjective

Let $f_k^0 : \mathcal{A}_k^0 \longrightarrow \mathcal{B}_k^0$, $\ker f_k^0 \leq \ker [n]_{\mathcal{A}_k^0}$ is finite

$\dim \mathcal{A}_k^0 = \dim \mathcal{A}_k = \dim A_K = \dim B_K = \dim \mathcal{B}_k = \dim \mathcal{B}_k^0$

$\Rightarrow f_k^0$ is surjective $\Rightarrow$ B has semistable (good) reduction.

Idea of proof:

- Let J be the Neron model of $J_0(N)$ over $\mathbb{Z}_N$

$J^0_{\mathbb{F}_N} \cong \mathrm{Pic}^0_{X_0(N)_{\mathbb{F}_N}/\mathbb{F}_N}$, where $X_0(N)$ is a minimal regular model for $X_0(N)$

(Raynaud)

Problem: The moduli problem for $X_0(p)$ is not representable (replace N by p from now on)

$X_0(p)$ over $\mathbb{Z}[1/p]$ is the coarse moduli space.

Construct a regular model of $X_0(p)$ over $\mathbb{Z}_p$ by the following way:

$$\begin{array}{c} X_0(p;\ell) \\ \parallel \\ GL_2(\mathbb{F}_\ell) \circlearrowright X_{\Gamma_0(p)\cap\Gamma(\ell)} \longleftarrow \text{representable over } \mathbb{Z}[1/\ell],\ \text{regular, flat (Katz-Mazur model)} \\ \big\downarrow \\ X_0(p) := X_{\Gamma_0(p)\cap\Gamma(\ell)}/GL_2(\mathbb{F}_\ell) \quad \text{Schematic quotient} \end{array}$$

We calculate the complete local ring at singular points ($\overset{p}{\text{supersingular}}$ ES)

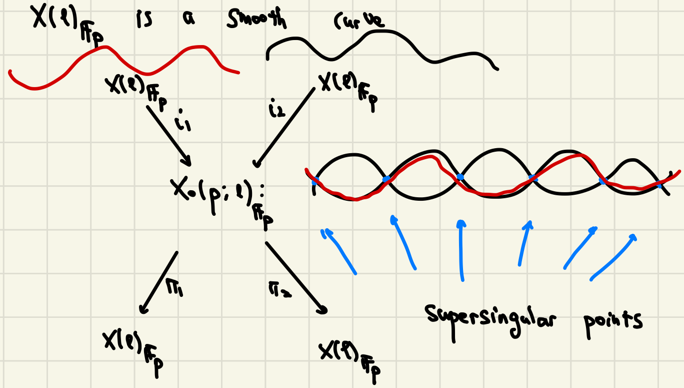
$\rightsquigarrow$ local ring of $X_0(p)$

After several steps of blow ups we get a regular model of $X_0(p)$

$\mathrm{Pic}^0_{X_0(p)/\mathbb{F}_p}$ is the collection of "degree 0" line bundles, which is a torus.

§ Geometry of $X_0(p;\ell)$

$X(\ell)_{\mathbb{F}_p}$ is a smooth curve



$i_1: (E, \Gamma(\ell)) \longmapsto (E \xrightarrow{F} E^{(p)}, \Gamma(\ell))$

$i_2: (E, \Gamma(\ell)) \longmapsto (E^{(p)} \xrightarrow{V} E, \Gamma(\ell)^{(p)})$

$\pi_1: (E, C, \Gamma(\ell)) \longmapsto (E, \Gamma(\ell))$

$\pi_2: (E, C, \Gamma(\ell)) \longmapsto (E/C, \overline{\Gamma(\ell)})$

$\pi_1 \circ i_1 = \pi_2 \circ i_2 = \mathrm{id},\ \pi_2 \circ i_1 = \pi_1 \circ i_2 = \mathrm{Frob}$

Prop (Deligne-Rapoport) (1) Let $x \in X_0(p;\ell)^{ss}_{\mathbb{F}_p}$, then $\widehat{\mathcal{O}}_{X_0(p;\ell)_{\mathbb{F}_p}, x} \cong k[[u,v]]/(uv)$, $k = k(x)$

(2) $\widehat{\mathcal{O}}_{X_0(p;\ell)_{\mathbb{Z}_p},x} \cong W(k)[[u,v]]/(uv-p)$

<u>Pf</u>: (1) Let $A = \widehat{\mathcal{O}}_{X_0(p;\ell)_{\bar{\mathbb{F}}_p},x}$, $C = \widehat{\mathcal{O}}_{X(\ell)_{\bar{\mathbb{F}}_p},x}$

- Consider $a = (i_1^*, i_2^*): A \longrightarrow C\times C$ is injective since A is reduced. $C\times C$ is the normalization of A.

  $S_0 + R_1 \longrightarrow$ CM since it's a quotient

  generically reduced

- $X(\ell)_{\bar{\mathbb{F}}_p}$ is smooth $\Rightarrow$ C is a DVR, let t be a uniformizer

$$\begin{cases} u = \pi_1^*(t) - \pi_2^*(t^p) \Rightarrow a(u) = (t - t^{p^2},\ t^p - t^p = 0) \\ v = \pi_2^*(t) - \pi_1^*(t^p) \Rightarrow a(v) = (t^p - t^p = 0,\ t - t^{p^2}) \end{cases}$$

$$\left.\begin{array}{l} a(u), a(v) \text{ generates } m_C \times m_C \\ a(u)\cdot a(v) = 0 \end{array}\right\} \Rightarrow$$

$$\begin{tikzcd} k[[u,v]]/(uv) \arrow[r, two heads] \arrow[rr, bend right, "\text{injective}"'] & A \arrow[r, hook] & C\times C \end{tikzcd} \Rightarrow k[[u,v]]/(uv) \cong A$$

(2) Let $R = \widehat{\mathcal{O}}_{X_0(p;\ell)_{\mathbb{Z}_p},x}$

then R is regular of dim 2, $R/pR \cong k[[u,v]]/(uv)$

$\Rightarrow R = W(k)[[u,v]]/(uv - pf)$ for some $f \in W(k)[[u,v]]$

$m_R = (p, u, v)$, $\dim_k m/m^2 = 2 \Rightarrow f$ is a unit (otherwise $\dim_k m/m^2 = 3$)

change coordinate $\Rightarrow R \cong W(k)[[u,v]]/uv-p$ $\square$

Now consider $\underset{\bar{G}}{\underset{\parallel}{GL_2(\mathbb{F}_\ell)/\{\pm1\}}} \curvearrowright X_0(p;\ell)_{\mathbb{Z}_p} \xrightarrow{\pi} X_0(p)_{\mathbb{Z}_p} := X_0(p;\ell)/\bar{G}$

take l s.t. $p \nmid |GL_2(\mathbb{F}_\ell)| = \ell(\ell-1)^2(\ell+1)$

Let x be a point on special fiber of $X_0(p)_{\mathbb{Z}_p}$

(1) $j(x) \neq 0, 1728$, $\bar{G}$ acts simply transitively on $\pi^{-1}(x)$

$\bar{G}$ permutes $\prod_{y \in \pi^{-1}(x)} \widehat{\mathcal{O}}_{X_0(p\ell),y} \Rightarrow \widehat{\mathcal{O}}_{X_0(p),x} \cong \widehat{\mathcal{O}}_{X_0(p\ell),y}$

- x ordinary. $\widehat{\mathcal{O}}_{X_0(p),x} \cong W[[u]]$ smooth point
- x ss. $\widehat{\mathcal{O}}_{X_0(p),x} \cong W[[u,v]]/uv-p$

(2) $j(x) = 1728$. $\mathrm{Stab}(y) = \mathbb{Z}/2\mathbb{Z}$ for $y \in \pi^{-1}(x)$

action: $u \longmapsto \zeta u$, $v \longmapsto \zeta^{-1} v$ $\zeta \neq 1$, $\zeta^2 = 1$

$$\widehat{\mathcal{O}}_{X_0(p),x} = \left(W[[u,v]]/(uv-p) \right)^{\mathbb{Z}/2\mathbb{Z}} = W[[u^2,v^2]]/u^2v^2 - p^2 \cong W[[x,y]]/(xy-p^2)$$

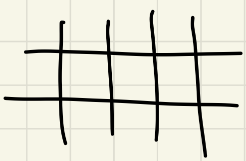
non-regular

(3) $j(x) = 0$. $\mathrm{Stab}(y) = \mathbb{Z}/3\mathbb{Z}$, similar action by $\zeta^3 = 1$, $\zeta \neq 1$

$$\Longrightarrow \widehat{\mathcal{O}}_{X_0(p),x} = W[[x,y]]/(xy-p^3)$$

do up to 2-steps of blow up at these points, we get a regular model $X_0(p)/_{\mathbb{Z}_p}$

<u>Prop</u>: Let C be the special fiber, then C is a reduced curve, whose components are $\mathbb{P}^1$, with node singularities.



$\mathbb{P}^1$ b/c $\underbrace{\Gamma(\ell)/\bar{G}}_{\text{the two components}} \cong \mathbb{P}^1$ via j-invariant. and blow-ups give some additional $\mathbb{P}^1$

§ relative Picard functor

$\pi: X \longrightarrow S$ proper flat, finite presented

- $\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T) / \pi^* \mathrm{Pic}(T)$
- If π admits a section $s: S \longrightarrow X$, then

$$\mathrm{Pic}_{X/S}(T) \cong \{ (\mathcal{L}, \lambda) : \mathcal{L} \in \mathrm{Pic}(X_T),\ \lambda: s^* \mathcal{L} \xrightarrow{\sim} \mathcal{O}_T \}$$

$$\mathcal{L} \longmapsto (\mathcal{L} \otimes \pi^* s^* \mathcal{L}^{-1},\ \lambda: s^* \mathcal{L} \otimes s^* \pi^* s^* \mathcal{L}^{-1} \cong s^* \mathcal{L} \otimes s^* \mathcal{L}^{-1} = \mathcal{O}_T)$$

- $\mathrm{Pic}_{X/S} \cong R^1 \pi_* \mathbb{G}_m$ for the fppf topology
- $\mathrm{Pic}^0_{X/S} \subset \mathrm{Pic}_{X/S}$ on geometric points degree 0 line bundles

__Thm__ (1) X/k proper curve. $\mathrm{Pic}^0_{X/k}$ is represented by a smooth k-gp scheme.

(2) If X is semistable, $\mathrm{Pic}^0_{X/k}$ is represented by a semi-abelian variety.

(3) $\mathrm{Pic}^0_{X_0(p)/\mathbb{Z}_p}$ represented by a smooth gp scheme over $\mathbb{Z}_p$ s.t. generic fiber $= J_0(p)$

__Cor__: NMP $\Longrightarrow$ $\mathrm{Pic}^0_{X_0(p)/\mathbb{Z}_p} \longrightarrow J^0_0(p)$ ← Néron model over $\mathbb{Z}_p$

$\rightsquigarrow$ $\mathrm{Pic}^0_{X_0(p)/\mathbb{F}_p} \longrightarrow J^0_0(p)_{\mathbb{F}_p}$, map between semi-AVs and Néron models

$\xrightarrow{\text{Raynaud}}$ isomorphism

ETS: $\mathrm{Pic}^0_{X_0(p)/\mathbb{F}_p}$ is a torus

Lemma Let $\pi : \tilde{X} \longrightarrow X$ normalization of a s.s. curve over k, then

$$0 \longrightarrow T \longrightarrow \mathrm{Pic}^0_{X/k} \xrightarrow{\pi^*} \mathrm{Pic}^0_{\tilde{X}/k} \longrightarrow 0$$

torus

AV

Our Case: $\widetilde{X_0(p)} = \sqcup \mathbb{P}^1$, $\mathrm{Pic}^0_{\widetilde{X_0(p)}/k} = \{0\} \Longrightarrow \mathrm{Pic}^0_{X_0(p)/\mathbb{F}_p}$ is a torus.

Pf of Lemma: $1 \longrightarrow \mathbb{G}_{m/X} \longrightarrow \pi_* \mathbb{G}_{m/X} \longrightarrow D \longrightarrow 0$

supported on singular points

$f : X \longrightarrow \mathrm{Spec}\, k$, $f_* D = \prod_{s \in S} \mathrm{Res}_{k(s)/k} \mathbb{G}_m$

LES $\rightsquigarrow$

$$1 \to \mathbb{G}_m \to \prod_{i \in I} \mathbb{G}_m \to \prod_{s \in S} \mathrm{Res}_{k(s)/k}(\mathbb{G}_m) \longrightarrow \mathrm{Pic}^0_{X/k} \longrightarrow \mathrm{Pic}^0_{\tilde{X}/k} \longrightarrow R^1 f_* D = 0$$

irred components of X

$\square$