Finishing up

Thm $N > 7$, $N \neq 13$ prime. Then $\nexists$ ell. curve $/\mathbb{Q}$ w/ rat'l pt. of order N.

Recall: Thm A : Suppose $\exists$ ab var $A/\mathbb{Q}$ w/ $f : X_0(N) \to A$ s.t.

1) A has good redn away from N
2) $A(\mathbb{Q})$ has rk 0
3) $f(0) \neq f(\infty)$.

Then $\nexists$ ell. curve $/\mathbb{Q}$ w/ rat'l pt. of order N.

Naturally, one looks for A among quotients of $J_0(N)$. Rmk: The condn. on N ensures that $J_0(N)$ is $\overset{\text{NOT}}{\wedge}$ trivial.

Thm B: $A/\mathbb{Q}$ ab. var., N, p $\overset{\text{dist.}}{\vee}$ primes, N odd. Suppose

a) A has good redn away from N
b) A has complete toric redn at N
c) The Jordan-Hölder factors of $A[p](\bar{\mathbb{Q}})$ ($G_\mathbb{Q} \circlearrowright$) are all trivial or cyclotomic.
"JH(p)"

Then $A(\mathbb{Q})$ has rk. 0.

So far, we

- defined $J_0(N)_{Eis} =: A$
- checked 1) = a) by properties of $X_0(N)$
    - b) by Jed
    - c) by Andrew, but w/ extra assumption
    - 3) by Andrew

Recall: $J_0(N) \underset{\text{isogeny}}{\sim} \prod_f A_f$, where f runs over Galois orbits of normalized wt 2 cuspidal eigenforms, $A_f := J_0(N)/\underset{\Pi}{P_f} J_0 N$ is of genus $[K_f : \mathbb{Q}]$.

Simplifying assumption: $[K_f : \mathbb{Q}] = 1$, $\forall f$.

Given a prime p, we want to pick the f's satisfying $JH(p)$. More precisely, f determines a map $\mathbb{T} \xrightarrow{f} \mathbb{Z}$ & $p_f = \ker(\mathbb{T} \xrightarrow{f} \mathbb{Z})$. then if $S^p = \{f$ s.t. A_f has $JH(p)\}$,

$$I^p = \bigcap_{f \in S^p} p_f$$

yields $A = J_0(N) / I J_0(N) =: J_0(N)_{\mathrm{Eis}}$.

$\underline{Q}$: Is S^p non-empty?

Recall: $[0] - [\infty] \in J_0(N)$ is a non-trivial torsion pt of order $|(N-1)$.

Claim: $\forall p \mid \mathrm{ord}([0]-[\infty])$ has $S^p \neq \phi$. So fix such a p.

Lem: $f \in S^p \Leftrightarrow p \mid a_\ell(f) - (\ell+1)$, $\forall \ell$

Def: The p-Eisenstein ideal is the ideal $\underset{\mathfrak{a}}{\wedge}$ generated by p & $T_\ell - (1+\ell)$.

We picked p s.t. $\mathbb{T}/p_f \xrightarrow{\sim} \mathbb{Z}$

$$T_\ell - (1+\ell) \longmapsto pk$$

the image of $\mathfrak{a}$ in $\mathbb{T}/p_f$ is NOT the unit ideal. $\Rightarrow$ $\mathfrak{a}$ is NOT the unit ideal

It is max. & $\mathbb{T}/\mathfrak{a} \simeq \mathbb{F}_p$. It satisfies $f \in S^p \Leftrightarrow p_f \subset \mathfrak{a}$.

$$\Rightarrow I^p = \bigcap_{\substack{\mathfrak{p} \subset \mathbb{T} \text{ min.} \\ \mathfrak{p} \subset \mathfrak{a}}} \mathfrak{p}.$$

In general, $T_p J_0(N) = \prod_{f, \lambda} V_{f,\lambda}$ $:=$ λ-adic Tate module of A_f, $\lambda \mid p$ place of K_f

- Pick the same p, $\mathfrak{a}$, I, and A. Claim $A(\mathbb{Q})$ has rk 0.
- Now, $S^p = \{f$ s.t. one of $V_{f,\lambda}$ has $JH(p)\}$

$A[p^n](\overline{\mathbb{Q}})$ has $JH(p)$

$\Updownarrow$

Review of Thm B

$\left\{\begin{array}{l} a) \\ b) \\ c) \end{array}\right. \Rightarrow A[p^n]$ is admissible $:=$ $\left\{\begin{array}{ll} \text{fin. flat away from } N & \text{i)} \\ \text{quasi-fin. ét. away from } p & \text{ii)} \\ \text{over } \mathbb{Z}[1/N]. \text{ It has fin by either } \mathbb{Z}/p\mathbb{Z} \text{ or } \mu_p & \text{iii)} \end{array}\right.$

Invariants for adm. gp schemes

- $h^i(G) =$ length of $H^i_{fppf}(\mathrm{Spec}\,\mathbb{Z}, G)$
- $\alpha(G) =$ # copies of $\mathbb{Z}/p\mathbb{Z}$ in G
- $\delta(G) = \log(\# G_{\mathbb{Q}}) - \log(\# G_{\mathbb{F}_N})$

We have $h^1(G) - h^0(G) \leq \delta(G) - \alpha(G)$. For $G = A[p^n]$, $\delta - \alpha$ is bdd indep. of n. Mordell-Weil Thm $\Rightarrow h^0(G)$ is also bdd indep. of n. $\Rightarrow$ so is $h^1(G)$.

Finally, $A(\mathbb{Q}) \hookrightarrow \varprojlim H^1_{fppf}(\mathbb{Z}, A[p^n])$
Kummer theory

1) Admissbility of G
2) compute of α, δ for G
3) use what G tells you about A to conclude.

<u>Claim</u> $A[p^n]_{\mathfrak{a}}$ is admissible.

completion of $\mathbb{T}$ at $\mathfrak{a}$ = direct factor of $\hat{\mathbb{T}}_p$

Note that $A(\mathbb{Q}) \otimes_{\mathbb{T}} \hat{\mathbb{T}}_{\mathfrak{a}} =: A(\mathbb{Q})_{\mathfrak{a}}$

We have $A(\mathbb{Q})_{\mathfrak{a}} \hookrightarrow \lim H^1_{fppf}(\mathbb{Z}, A[p^n]_{\mathfrak{a}})$.

<u>Claim</u>: $A(\mathbb{Q})_{\mathfrak{a}}$ fin $\Rightarrow A(\mathbb{Q})$ is fin.

$A[p^n]_{\mathfrak{a}} \subset A[\mathfrak{a}^m]$ for some m. ISTS: $A[\mathfrak{a}^m]$ has $JH(p)$.

$$0 \to A[\mathfrak{a}] \to A[\mathfrak{a}^m] \to A[\mathfrak{a}^{m-1}]^{\oplus k}$$

$\Rightarrow$ by induction, we need to prove that $A[\mathfrak{a}]$ has $JH(p)$

- Eichler-Shimura $\Rightarrow$ Frob$_\ell$ has ann. poly. $(X-1)(X-\ell)$ $\Rightarrow$ gen. e-values of Frob$_\ell$ are only 1 & ℓ.
- $V = A[\mathfrak{a}]$, $W = V \oplus V^\vee(1)$.

The 1-gen. e-space of Frob$_\ell$ on W has the same dim as the ℓ-gen. e-space. $\Rightarrow$ $(X-1)^n(X-\ell)^n$.