Toric reduction

for rk 0

Thm^A, Thm^B: Existence of a special a.v. $\Rightarrow \nexists$ N-tors. pt. + criterion

- N prime $X_0(N) \to A$ rk 0
- we have AJ: $X_0(N) \to J_0(N) \to A$

To prove Thm.^{B}, we need: $J_0(N)$ has good redn. outside N. ✓ bc. $X_0(N)$ has a sm. model / $\mathbb{Z}[1/N]$.

Today's goal: $J_0(N)$ has toric redn. at N.

(good)

Rmk: This is sufficient since toric redn. passes through isogeny.

Pf: A, B avs / DVR R k = res field, $K = \mathrm{Frac}(R)$. If

$$A \to B$$ ← isogeny cat. is semi-simple

isogeny / K, then $\exists$ $B \to A$ s.t. $f \circ g = [n]$.

- $\mathcal{A}, \mathcal{B}$ Néron model of A, B over R
- NMP $\Rightarrow$ f, g extends to $\mathcal{A}$ & $\mathcal{B}$ & $f \circ g = [n]$.

Now,

$$\overline{[n]} = \bar{f} \circ \bar{g} : \mathcal{A}_k \to \mathcal{A}_k \text{ surj.}$$

Consider $\bar{f}_k^\circ : \mathcal{A}_k^\circ \to \mathcal{B}_k^\circ$. $\ker \bar{f}_k^\circ \subset \ker \overline{[n]}_k$ fin. And
$\dim \mathcal{A}_k^\circ = \dim \mathcal{A}_k = \dim A_K = \dim B_K = \dim \mathcal{B}_k^\circ$ $\Big\} \Rightarrow$

$\bar{f}_k^\circ$ is surj.

$\square$

(switch from N to p)

Idea of the proof: Let $\mathcal{J}$ be the Néron model of $J_0(p)$ over $\mathbb{Z}_p$.

Fact (Raynaud): $\mathcal{J}^\circ_{\mathbb{F}_p} \simeq \mathrm{Pic}^\circ_{X_0(p)_{\mathbb{F}_p}/\mathbb{F}_p}$ where $X_0(p)$ is a semi-stable model of $X_0(p)_{\mathbb{Q}_p}$; $X_0(p)_{\mathbb{F}_p}$ the special fiber of $X_0(p)$.

Problem: The moduli problem for $X_0(p)$ is not representable, so we use the coarse moduli space. [Katz – Mazur]

Consider $X_{\Gamma_0(p) \cap \Gamma(\ell)}$ ← representable / $\mathbb{Z}[1/\ell]$, reg. flat

$$\begin{array}{c} X_{\Gamma_0(p)\cap\Gamma(\ell)} \circlearrowleft GL_2(\mathbb{F}_\ell) \\ \downarrow \\ X_0(p) = X_{\Gamma_0(p)\cap\Gamma(\ell)} / GL_2(\mathbb{F}_\ell) \end{array}$$

$\ell \neq p$ another prime

schematic quotient

We will compute the completed local ring at supersingular pts on $X_0(p)_{\mathbb{F}_p}$

⇝ After taking some blow ups, we get a semi-stable model for $X_0(p)$.

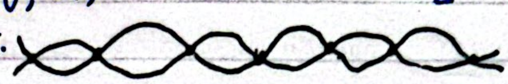
$\mathrm{Pic}^0_{X_0(p)_{\mathbb{F}_p}/\mathbb{F}_p}$ "≈" deg 0 line bundles. ⇝ prove toric redn by understanding the geometry of $X_0(p)_{\mathbb{F}_p}$.

§ Geometry of $X_0(p;\ell)$

$E \xrightarrow{F} E^{(p)}$

$(E, \operatorname{Ker} F, \cdots)$ $\quad$ $(E, \operatorname{Ker} V, \cdots)$

$F = $ Frob $\quad X_0(p;\ell)_{\mathbb{F}_p}$ $\quad V = $ Ver.



$X_0(\ell)$

$X_0(\ell)$

i_1 $\quad$ i_2

supersingular locus $\quad E^{(p)} \xrightarrow{V} E$

$X_0(\ell)$ $\quad$ $X_0(\ell)$ $(E, P(\ell)\text{-str.})$

$(E, P(\ell)\text{-str.})$

$(E, C, P(\ell)\text{-str.})$

$$\begin{tikzcd} & X_0(p;\ell) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_0(\ell)_{\mathbb{F}_p} & & X_0(\ell)_{\mathbb{F}_p} \end{tikzcd}$$

$(E, P(\ell)\text{-str.})$ $\quad$ $(E/C, \overline{P(\ell)}\text{-str.})$

We have:

- $\pi_1 \circ i_1 = \pi_2 \circ i_2 = \mathrm{id}$
- $\pi_2 \circ i_1 = \pi_1 \circ i_2 = $ Frob

Prop (Deligne – Rapoport):

1) Let $x \in X_0(p,\ell)^{ss}_{\mathbb{F}_p}$, then $\hat{\mathcal{O}}_{X_0(p;\ell),x} \simeq k[\![u,v]\!]/(uv)$ where $k =$ residue field at x

2) $\hat{\mathcal{O}}_{X_0(p;\ell)_{\mathbb{Z}_p},x} \simeq W(k)[\![u,v]\!]/(uv-p)$.

Pf: 1) Denote $A = \hat{\mathcal{O}}_{X_0(p;\ell)_{\mathbb{F}_p},x}$ $\quad$ t uniformizer

$C = \hat{\mathcal{O}}_{X_0(\ell)_{\mathbb{F}_p},x}$ (DVR) $\quad$ (x = preimage of x on $X_0(\ell)_{\mathbb{F}_p}$)

Consider $a := (i_1^*, i_2^*)\ A \to C \times C$

- a is inj. because A is <u>reduced</u> & a is the normalization map

$\Uparrow$ $X_0(p;\ell)_{\mathbb{Z}_p}$ is reg. ⇒ A is Cohen-Macaulay

⇒ reduced by Serre's thm, …

"$S_0 + R_1$"

$\Uparrow$ gen. red. $\quad$ $\Uparrow$ CM

- $u = \pi_1^*(t) - \pi_2^*(t^p)$
$v = \pi_2^*(t) - \pi_1^*(t^p)$ $\in A$

also a uniformizer in C

- $a(u) = (t - t^{p^2}, t^p - t^p = 0)$
$a(v) = (t^p - t^p = 0, t - t^{p^2})$ ⇒ $a(u), a(v)$ generates $m_C \times m_C$, $a(u) \cdot a(v) = 0$

⇒ $k[\![u,v]\!]/(uv) \twoheadrightarrow A \hookrightarrow C \times C$ ⇒ $k[\![u,v]\!]/(uv) \simeq A$.

inj ②

2) $R = \hat{\mathcal{O}}_{X_0(p,\ell)_{\mathbb{Z}_p}, x} \overset{KM}{\Rightarrow} R$ is reg. of dim 2 & 1) $\Rightarrow R/pR \simeq k[\![u,v]\!]/(uv)$

$\Rightarrow W(k)[\![u,v]\!]/(uv - pf) \xrightarrow{\sim} R$ & iso. by dim.

reg. $\Rightarrow \dim_k m/m^2 \simeq 2$, $m = (u,v,p) \Rightarrow f$ is a unit (o/w, $\dim_k m/m^2 = 3$)

$\Rightarrow R \simeq W(k)[\![u,v]\!]/(uv-p)$. $\square$

$\bar{G} := GL_2(\mathbb{F}_\ell) \overset{(ss)}{\curvearrowright} X_0(p;\ell)_{\mathbb{Z}_p}$. Take ℓ s.t. $p \nmid |GL_2(\mathbb{F}_\ell)| = \ell(\ell-1)^2(\ell+1)$

then $\{\pm 1\}$ the quotient exists.

$$\begin{array}{c} X_0(p;\ell)_{\mathbb{Z}_p} \supset \pi^{-1}(x) \\ \downarrow \pi \\ X_0(p)_{\mathbb{Z}_p} \ni x \end{array}$$

1) $j(x) \neq 1728, 0$. then $\bar{G}$ acts simply transitively on $\pi^{-1}(x) \Rightarrow$ $\bar{G}$ permutes $\prod_{y \in \pi^{-1}(x)} \hat{\mathcal{O}}_{X_0(p;\ell)_{\mathbb{Z}_p}, y}$

$$\Rightarrow \hat{\mathcal{O}}_{X_0(p), x} = \Big(\prod_{y \in \pi^{-1}(x)} \hat{\mathcal{O}}_{X_0(p;\ell)_{\mathbb{Z}_p}, y}\Big)^{\bar{G}} = \hat{\mathcal{O}}_{X_0(p;\ell)_{\mathbb{Z}_p}, y}$$

2) $j = 1728$ $y \in \pi^{-1}(x)$. $\mathrm{Stab}(y) = \mathbb{Z}/2\mathbb{Z} \curvearrowright \hat{\mathcal{O}}_{X_0(p;\ell), y} = W(k)[\![u,v]\!]/(uv-p)$

The action is given by

$$\begin{cases} u \mapsto \zeta u \\ v \mapsto \zeta v \end{cases} \quad \zeta^2 = 1,\ \zeta \neq 1$$

$$\Rightarrow W(k)[\![u,v]\!]/(uv-p)^{\mathbb{Z}/2\mathbb{Z}} \simeq W(k)[\![u^2,v^2]\!]/(u^2v^2 - p^2) \simeq W(k)[\![u,v]\!]/(uv - p^2)$$

3) $j = 0$ Similar computations show that

$$\hat{\mathcal{O}}_{X_0(p), x} \simeq W(k)[\![u,v]\!]/(uv - p^3)$$

In case 2) & 3), the local rings are not reg. $\underset{\text{blow-up}}{\overset{2\text{-step}}{\rightsquigarrow}} X_0(p)_{\mathbb{Z}_p}$ reg. w/ semi-stab. reduction.

Prop: $X_0(p)_{\mathbb{F}_p}$ is a reduced curve whose components are $\mathbb{P}^1$ and w/ nodal singularities.

(Rmk: $\mathbb{P}^1$ appears because $X_0(\ell)/\bar{G} \simeq \mathbb{P}^1$ via the j-inv.)

§ Relative Picard functor

- $\pi: X \to S$ proper, flat, fin-presented
- $\mathrm{Pic}_{X/S}(T) := \mathrm{Pic}(X_T)/\pi^*\mathrm{Pic}(T) \simeq R^1\pi_* \mathbb{G}_m$ for fppf top.

- $\mathrm{Pic}^\circ_{X/S} \subset \mathrm{Pic}_{X/S}$ taking deg 0 line bdles on geometric pts

<u>Prop</u>: 1) X/k proper curve, then $\mathrm{Pic}^\circ_{X/k}$ is rep. by a sm. k-gp. sch.
2) If X is semi-stab., then $\mathrm{Pic}^\circ_{X/k}$ is rep by a semi-ab. var.
3) $\mathrm{Pic}^\circ_{X_0(p)/\mathbb{Z}_p}$ is represented by a smooth gp sch. $/\mathbb{Z}_p$ w/ generic fiber iso. to $J_0(p)$.

<u>Cor</u>: NMP $\Rightarrow$ $\mathrm{Pic}^\circ_{X_0(p)/\mathbb{Z}_p} \to \mathcal{J}_0^\circ(p)$ Néron model of $J_0(p)/\mathbb{Z}_p$

Redn mod p $\Rightarrow$ $\underbrace{\mathrm{Pic}^\circ_{X_0(p)_{\mathbb{F}_p}/\mathbb{F}_p}}_{\text{semi-ab.var.}} \xrightarrow{\sim} \mathcal{J}_0^\circ(p)_{\mathbb{F}_p}$ iso. by Raynaud's thm

<u>Goal</u>: $\mathrm{Pic}^\circ_{X_0(p)_{\mathbb{F}_p}/\mathbb{F}_p}$ is a torus.

<u>Lem</u>: Let $\pi: \tilde{X} \to X$ be a normalization of a s.s. curve X. Then

$$0 \to \underbrace{T}_{\text{torus}} \to \mathrm{Pic}^\circ_{X/k} \xrightarrow{\pi^*} \underbrace{\mathrm{Pic}^\circ_{\tilde{X}/k}}_{AV} \to 0$$

(In our case, $\tilde{X} = \sqcup \mathbb{P}^1$

$$0 \to T \to \mathrm{Pic}^\circ_{X_0(p)/\mathbb{F}_p} \to 1 \to 0.$$

$\Rightarrow$ we win.)

<u>Pf of Lem</u>: $1 \to \mathbb{G}_{m,X} \to \mathbb{G}_{m,\tilde{X}} \to D \to 1$. Write $f: X \to k$
(singular pts) $=$ subset supp. in 0-dim

Apply Rf_* $\Rightarrow$ index by irred components of X

$$1 \to \mathbb{G}_m \to \prod_{i\in I} \mathbb{G}_m \to f_* D \to \mathrm{Pic}^\circ_{X/k} \xrightarrow{\pi^*} \mathrm{Pic}^\circ_{\tilde{X}/k} \to 1$$

<u>Fact</u>: $f_* D = \prod_{s \in S} \mathrm{Res}_{k(x)/k} \mathbb{G}_m$ (S = singular locus)

$$\Rightarrow 1 \to \underbrace{T}_{\text{torus}} \to \mathrm{Pic}^\circ_{X/k} \to \mathrm{Pic}^\circ_{\tilde{X}/k} \to 1$$

$\square$.