Talk 3 Abelian varieties, Jacobians, & Their Néron Models

Naina Praveen

Plan:

1) minimal reg. models + NOS
2) Jacobians
3) Néron models + NOS

e.g. $E/\mathbb{Q}_p$ $\{y^2 = x^3 + p^6\} \simeq \{y^2 = x^3 + 1\}$

$x \mapsto p^2 x'$ different behavior mod p

$y \mapsto p^3 x'$

work w/ models

Notation:

- K loc. field
- $\mathcal{O}_K$ ring of int
- k res. field

"nice": sm., proj., geom. int.

Def: A model of a nice curve C/K is an $\mathcal{O}_K$-sch. $\mathcal{C}$ s.t. it's of fin. type, flat, proper & equipped w/

$$\mathcal{C} \times_{\mathcal{O}_K} K \simeq C$$

generic fiber $: \mathcal{C} \times_{\mathcal{O}_K} K$

special fiber $: \mathcal{C} \times_{\mathcal{O}_K} k =: \mathcal{C}_k$

e.g. $E/\mathbb{Q}_p$ ell. curve. w/ min. Weierstrass eqn.

$$y^2 = x^3 + ax + b \qquad a, b \in \mathbb{Z}_p$$

Consider

$$\mathcal{E}: \quad y^2 z - x^3 - axz^2 - bz^3 = 0$$

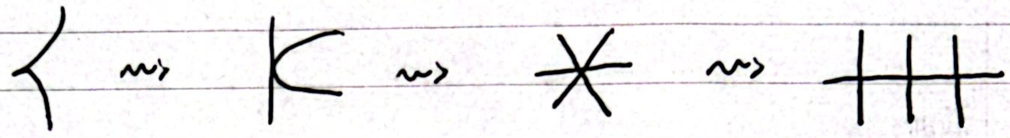
special fiber $\bar{E}$

If $v_p(\Delta) < 12$, this is called a minimal Weierstrass model

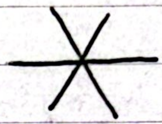
3 possibilities

1) $\bar{E}/\mathbb{F}_p$ is an ell. curve $\Leftrightarrow v_p(\Delta) = 0$ "good redn"
2) $\bar{E}/\mathbb{F}_p$ has a node $\Leftrightarrow x^3 + ax + b$ mod p has a unique double root "multi."
3) — " — cusp $\Leftrightarrow$ — " — triple root "additive"

e.g. 1) $y^2=x^3+p$ ⇝ cusp mod p ⇝ blow-ups



e.g. 2) $y^2=x^3+p^2$ has min. reg model given by



which has an ord. triple pt.

<u>Two ways</u> (to pick a perferred model)

1) require $\mathcal{E}/\mathcal{O}_K$ to be reg.
2) singularities "not too bad" (semi-stab.)

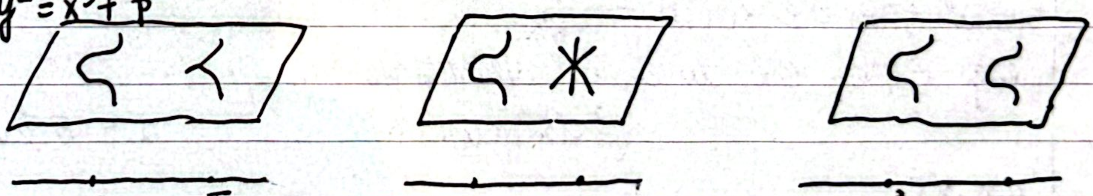
- 1) always exists & $\mathcal{E}$ is <u>minimal</u> if $\forall$ reg. model $\mathcal{E}'$, $\exists\ \mathcal{E}'\to\mathcal{E}$ extends the identity $/K$.

  <u>Thm</u>: Minimal reg. model exists & is unique.
- 2) ? special fibers can be quite complicated (see e.g. 2))

  <u>Def</u>: A curve X is <u>semi-stab.</u> if it is red. & all its singular pts are ordinary double pt.

  <u>Thm</u> (Semistab. redn thm): There is always a fin. extn. K'/K where E attains semistable redn.

e.g. $y^2=x^3+p^2$



$\mathcal{O}_p$ $\mathbb{F}_p$ original model

$\mathcal{O}_p$ $\mathbb{F}_p$ min. reg

$\mathcal{O}_p[\sqrt[3]{p}]$ $\mathbb{F}_p$ $\mathbb{Z}_p[\sqrt[3]{p}]$ semi-stab mod.

- $\mathcal{E}=\mathrm{Jac}(E)$ gp sch.
- $T_\ell(E)=\varprojlim_n E[\ell^n]$

<u>Thm</u> (Néron-Ogg-Shafarevich):

- E has good redn $\Leftrightarrow$ $T_\ell(E)$ is unr.
- E has semi-stab redn $\Leftrightarrow$ I_K acts unipotently

Jacobians

- K any field
- X sm. proj. conn. curve
- $\mathrm{Pic}(X)$ = gp of iso. cls. of line bdls of X

  $\mathrm{Pic}^0(X)$ = subgp —"— deg 0 —"—

Def: A family of elts on Pic^0 / T is a line bdl $\mathcal{L}$ on $X_T = X \times T$ s.t. $\mathcal{L}|_{X \times \{t\}}$ has deg 0, $\forall t \in T$.

- $\mathcal{F}(T) := \{$ set of iso. cls. of families $/ T \}$

<u>NOT</u> rep'le in general: Suppose $\mathcal{F}$ is rep. by J. L any l.b. on T, consider

$$p : X_T = X \times T \to T$$

we have $p^*(L)|_{X \times \{t\}} \simeq \mathcal{O}_X$, $\forall t \in T \Rightarrow \deg 0 \Rightarrow \exists f : T \to J$ s.t. $p^*(L) \simeq f^*(\mathcal{L})$ w/ $\mathcal{L}$ some univ. bdl on J, $p^*(L)$ triv. on fiber $\Rightarrow f$ maps T to the same pt. $\Rightarrow f^*(\mathcal{L})$ triv. $\lightning$

Def: $\mathcal{G}(T) := \mathcal{F}(T) / p^* \mathrm{Pic}^0(T)$. [Again, $\mathcal{G}$ is <u>NOT</u> rep'le in gen. But, when $X(K) \neq \phi$, then $\mathcal{G}$ is a sheaf.] ($\mathcal{F}$ not nece. a sheaf)

Let $x \in X(K)$. Put $G_x(T)$ cat. of pairs (L, i) where L is a line bdl on $X \times T$ & $i : L_{\{x\} \times T} \xrightarrow{\sim} \mathcal{O}_T$. Define

$$\mathcal{G}_x : T \mapsto \{ \text{iso. cls. of } G_x(T) \}$$

then $\mathcal{G}_x \simeq \mathcal{G}$.

<u>Thm</u> Suppose X has a rat'l pt, then the sheaf $\mathcal{G}$ is rep. by $\mathrm{Jac}(X)$.

- Explicit description
- $X^{(g)}$ symmetric power space of X pts on $X^{(g)} \leftrightarrow$ eff. divisors on X of deg g
- D, D' divisors of deg g.

$$[D + D' - g[x]] = D'' \ ?$$

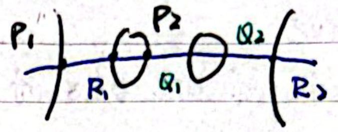
- $\ell(D)$ = dim. of v.s. of rat'l fcns assoc. to D
- RR: $\ell(D) \geq \deg(D) - g + 1 = 1$
- semi-continuity result $\rightsquigarrow$ $U := \{ (D, D') \in X^{(g)} \times X^{(g)} : \ell(D + D' - g[x]) = 1 \}$ is open & non-empty.
- get rat'l gp law: $X^{(g)} \times X^{(g)} \to X^{(g)}$

<u>Weil</u>: This rat'l gp law is actually a gp law, i.e., $\exists !$ gp var. J w/ a birational gp. hom. $X^{(P)} \dashrightarrow J$.

J rep. $\mathcal{G}$.

• $g=1$ ell. curve, recovers the usual gp law

• $g=2$ $y^2=f(x)$ deg $f=5$ or 6 $X^{(2)} \leftrightarrow$ pair of pts



Néron models

Thm: Let A/K be an ab. var. Then $\exists$ a sep., fin. type sm. gp sch. $\mathcal{A}/\mathcal{O}_K$ w/ gen. fiber A satisfying the univ. property "Néron mapping property" (NMP) that, for each sm. $\mathcal{O}_K$-sch. $\mathcal{Y}$, any K-morphism

$$\mathcal{Y}_K \to A$$

extends uniquely to an $\mathcal{O}_K$-mor.

$$\mathcal{Y} \to \mathcal{A}$$

Rmk:

1) NMP: $\mathrm{Hom}_{\mathcal{O}_K}(\mathcal{Y}, \mathcal{A}) = \mathrm{Hom}_K(\mathcal{Y}_K, A)$. Take $\mathcal{Y} = \mathrm{Spec}\, \mathcal{O}_K$, then $\mathcal{A}(\mathcal{O}_K) = A(K)$

2) Unlike curves, we replace "properness" w/ "smoothness"

3) Néron models do NOT commute w/ base change. More precisely, if K'/K is a fin extn. & $\mathcal{A}$ (resp. $\mathcal{A}'$) Néron model $/K$ (resp. K'). Then

$$\mathcal{A} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \xrightarrow{f} \mathcal{A}'$$

is a sm. sch. w/ gen. fiber $A_{K'}$ & f is not an iso. unless K'/K is unr.

Thm For an ell. curve, the Néron model $\mathcal{E}$ is the sm. locus of the min. reg. model.

$\mathcal{A}$ = Néron model of A

Def The redn. of an ab. var. $/K$ is the gp. var. $\tilde{A}_k = \mathcal{A} \otimes_{\mathcal{O}_K} k$. If $\tilde{A}_k$ is an ab. var. then we say that A/K has good redn.

The id. component of the Néron model $\mathcal{A}^0$, is the open subsch. whose special fiber is the conn component of $\tilde{A}_k$ (i.e., remove the cl. subset consisting of the union of (fin. many) $\tilde{A}_k^0$ components of the special fiber not containing the id. component.

Since NMP $\Rightarrow$ $\mathcal{A}(\mathcal{O}_K) = A(K)$, we have redn hom $A(K) \to \tilde{A}_k(k)$

Denote by $A_0(K)$ the pts in $A(K)$ red. to $\tilde{A}_k^0(k)$. The gp.

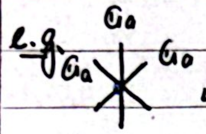
$$A(K)/A_0(K)$$

is fin. & its order is called the Tamagawa number.

Alternatively, $\Phi := A_k / A_k^0$ is a fin. ét. gp sch. $/k$ & $\exists$ SES

$$0 \to A_k^0 \to A_k \to \Phi \to 0$$

↳ gp of conn. components

e.g.  $\Phi = \mathbb{Z}/3\mathbb{Z}$ ⇝ $y^2 = x^3 + p^2$

 $\Phi = \{0\}$ ⇝ $y^2 = x^3 + p$

Chevalley: $0 \to T \times U \to A_k^0 \to B \to 0$

- B is a k-ab. var.
- T alg. torus
- U unipotent alg gp.

Def: A/K has

- good redn. if $U, T = 0$
- semistab. redn. if $U = 0$
- (purely) toric redn. $U, B = 0$
- split-toric redn. if $U, B = 0$ & T is split.

Thm (NOS):

1) A has good redn $\iff T_\ell A$ is unr.
2) A has semi-stab redn. $\iff I_K$ acts trivially on $T_\ell(A)$

Rmk: 1) For a curve C, $\mathrm{Jac}(C)$ has semi-stab redn. $\iff C$ is.

2) Even if A/K is semi-stab., the Néron model does not commute w/ ram. base change. ; but $\mathcal{A}^0$ commutes w/ base change.