Talk 2 Group schemes

Shin Thant

Def: S scheme. a gp scheme $/S$ is a sch. $G \to S$ equipped w/

$m: G \times_S G \to G$

$i: G \to G$

$e: S \to G$

w/ certain axioms. e.g. take S-sch. T then let $G(T) = \mathrm{Hom}_S(T, G)$, we get

$$m: \underset{\mathrm{Hom}_S(T, G\times_S G)}{\overset{\shortparallel}{G(T) \times G(T)}} \to \underset{\mathrm{Hom}_S(T,G)}{\overset{\shortparallel}{G(T)}}$$

$i: G(T) \to G(T)$, $e \in G(T)$

We require each $G(T)$ to be a gp. Write $G(R) = G(\mathrm{Spec}\, R)$

e.g. k field

- $\mathbb{G}_a = \mathrm{Spec}\, k[t]$ w/ $m: \mathrm{Spec}\, k[x,y] \to \mathrm{Spec}\, k[t]$, $t \mapsto x+y$
  $i: \mathrm{Spec}\, k[t] \to \mathrm{Spec}\, k[t]$, $t \mapsto -t$
  $e:$ — " — $t \mapsto 0$

$\Rightarrow G(R) = (R, +)$ for any k-alg. R

- $\mathbb{G}_m(R) = (R^\times, \times)$ w/ underlying sch. $k[t, t^{-1}]$
- $\mu_n(R) = \{x^n = 1, x \in R\}$ — " — $k[t]/(t^n - 1)$
- const. gp. sch. $\underline{\Gamma}$ w/ arbitrary gp. Γ
  $\underline{\Gamma}(R) = \{$ loc. const. fcns on $\mathrm{Spec}\, R$ w/ values in $\Gamma\}$

Important e.g. $\mathbb{Z}/n\mathbb{Z}$

& $\mu_n \neq \mathbb{Z}/n\mathbb{Z}$ in general [$k = \mathbb{Q}$, then $\mu_n(\mathbb{Q}) \neq \mathbb{Z}/n\mathbb{Z}(\mathbb{Q})$ $n > 2$]

- A ab. var. $/k$, kernel of $[n]: A \to A$ is a gp sch. $A[n]$.

Def: G/S is fin. flat if it is commutative & $G \xrightarrow{f} S$ is fin. flat. $f_*(\mathcal{O}_G)$ is loc. free $\mathcal{O}_S$-mod. of const. rk r. ← "order" of G.

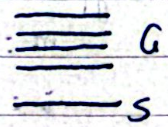
e.g. $S = \mathrm{Spec}\, k$, $G = \mu_n$ then $f_*(\mathcal{O}_{\mu_n}) = k[t]/(t^n - 1)$ has rk n / k. (but $|\mu_n(\bar{k})| \neq n$ if $\mathrm{char}(k) \mid n$.)

• E/k ell. curve, then $E[p]$ has order p^2

Def: A fin. ét. gp. sch. is a fin. flat gp. sch. G/S w/ étale structure morphism

e.g. • const. gp sch. is always ét.

• In char 0, every fin. flat gp sch. is ét.



• μ_p is not ét. / $S = \operatorname{Spec} k$, $\operatorname{char}(k) = p$.
$\because \operatorname{Spec} k[t]/(t^p - 1) = \operatorname{Spec} k[t]/(t-1)^p$
not separable.

Thm: Fix k^{sep}/k, then $\exists$ equiv. of cats

$$\{\text{fin. ét. gp. sch.}/k\} \longleftrightarrow \{\text{fin. ab. gp. w/ cts } G_k\text{-action}\}$$
$$G/k \longmapsto G(k^{sep}) \circlearrowleft G_k$$

Can gen. to a conn. base sch. S by replacing the Galois gp. w/ ét. fund gp. of any geom. pt.

Thm G/S fin. flat. of order n which is inv. on S, then G is ét.

⌜Sketch of proof: reduce to the case where $S = \operatorname{Spec} \bar{k}$. In this case, ét $\Leftrightarrow$ $T_e G = 0$.

Deligne: multn. by $n = 0$ on G $\Rightarrow$ $[n]: T_e G \to T_e G$ is zero. but this is just $\times n$ again. $\Rightarrow T_e G = 0$ ⌟

Def: G/S fin. flat. The Cartier dual of G is a fin. flat gp. sch. $G^\vee/S$ that satisfies

$$G^\vee(T) = \operatorname{Hom}_{T\text{-Grp}}(G_T, \mathbb{G}_{m,T})$$

e.g. 1) $\mu_n^\vee = \mathbb{Z}/n\mathbb{Z}$

2) A ab. var. $A[n]^\vee = A^\vee[n]$

Let $K/\mathbb{Q}_p$ be a fin. extn., $\mathcal{O}_K$ ring of ints. $k = \mathcal{O}_K/m$. e ramification ~~index~~ deg.

Def: A prolongation of a fin flat gp sch. G_0/K is a fin. flat. gp. sch. $G/\mathcal{O}_K$ s.t. $G_K \xrightarrow{\sim} G_0$

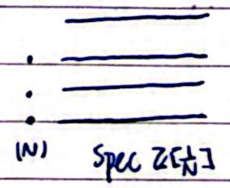
Thm (Raynaud): If $e < p-1$, then any two prolongation are iso.

e.g. $K = \mathbb{Q}_p(\zeta_p)$, then $e = p-1$. In this case $\mathbb{Z}/p\mathbb{Z}$ & $\mu_p/\mathcal{O}_K$ both have

the same gen. fiber $(\mathbb{Z}/p\mathbb{Z})_K \simeq (\mu_p)_K$ since K contains all p-th roots of 1.
But $(\mathbb{Z}/p\mathbb{Z})_R \not\simeq (\mu_p)_R$.

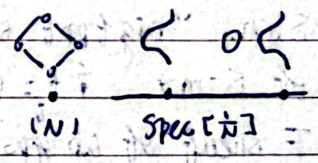
Admissible gp sch. N prime. $p \neq N$ another prime

Def: A pre-adm. gp. sch. $G/S = \mathbb{Z}[\frac{1}{N}]$ is flat & killed by a power of p.
A gp. sch. $G/\mathbb{Z}$ is pre-adm. if it is quasi-fin. sep., flat / $\mathbb{Z}$ & resn. to Spec $\mathbb{Z}[\frac{1}{N}]$ is fin. flat.



Context: ab. var. $A/\mathbb{Q}$ w/ good redn. outside N. Then let A be the Néron model over $\mathbb{Z}$. $A^\circ[p]$ is pre-adm.

e.g. $A = E$ ell. curve w/ mult. bad redn at N



Def: $G/\mathbb{Z}[\frac{1}{N}]$ pre-adm. An adm. filn. is a filn.

$$0 = \mathrm{Fil}^0 G \subset \mathrm{Fil}^1 G \subset \cdots \subset \mathrm{Fil}^n G = G$$

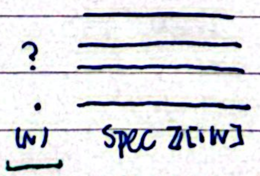
by cl. subgp. schs s.t. each gr. piece is iso. to $\mathbb{Z}/p\mathbb{Z}$ or μ_p. G is adm if it has such a filn.

$G/\mathbb{Z}$ is adm if $G/\mathbb{Z}[\frac{1}{N}]$ is adm.

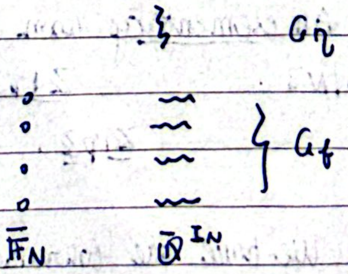
Fix $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_N$, get inertia $I_N \subset G_{\mathbb{Q}}$.

Thm. Let G' be a pre-adm. gp sch. / $\mathbb{Z}[\frac{1}{N}]$. Then

$$\{G/\mathbb{Z} \text{ pre-adm. ext. } G'\} \longleftrightarrow \{\text{sub } G_{\mathbb{Q}_p}\text{-mod. of } G(\bar{\mathbb{Q}})^{I_N}\}.$$



$G_R = G_f \sqcup G_\eta$
where G_f is fin. ét./Spec R;
G_η has $\varnothing$ special fiber



Spec R $\to$ Spec $\mathbb{Z}[1/N]$

$R :=$ preimage of the strict Henselization of $\mathbb{Z}_{(N)}$ in $\bar{\mathbb{Q}}_N$ $\cap$ $\bar{\mathbb{Q}}$ Frac$(R) = \bar{\mathbb{Q}}^{I_N}$
res. field $\bar{\mathbb{F}}_N$

$$G_f(\bar{\mathbb{Q}}^{I_N}) \twoheadrightarrow G_f(\bar{\mathbb{F}}_N)$$

$\exists$ a bij. $\quad G'(\bar{\mathbb{F}}_N)$

$$\phi: G(\bar{\mathbb{F}}_N) \to G_f(\bar{\mathbb{Q}}^{I_N}) \to G(\bar{\mathbb{Q}})^{I_N}$$

$H := \mathrm{Im}(\phi)$.

- If H is triv., then "extn by zero", denoted by G^b
- If H is max., then get $G^\#$ $\quad \overset{\shortparallel}{G(\bar{\mathbb{Q}})^{I_N}}$

Cor: G' has a fin. extn. $\Leftrightarrow$ $G'(\bar{\mathbb{Q}})^{I_N} = G'(\bar{\mathbb{Q}})$.

~~e.g~~

Invariants

- $e(G) = |G_{\mathbb{R}}|$
- $\delta(G) = \log_p |G_{\mathbb{R}}| - \log_p |G_{\mathbb{F}_N}|$
- $\alpha(G) = \#\ \mathbb{Z}/p\mathbb{Z}$ appearing in the adm. filn.
- $h^i(G) = \log_p |H^i_{fppf}(\mathrm{Spec}\ \mathbb{Z}, G)|$

=

- S sch. "big fppf site $/S$" = cat. of S-sch. where the coverings are families of S-mor. $\{f_i: U_i \to U\}_{i\in I}$ w/ f_i flat, loc. of fin. presentation.
- $\mathcal{F}$ sheaf of ab gps on S_{fppf}
- $H^i_{fppf}(S, \mathcal{F})$, $i \geq 0$

  e.g. $H^0_{fppf}(S; \mathcal{F}) = \mathcal{F}(S)$

  $H^1 \Leftrightarrow$ fppf-torsor

(If G/S is a comm. gp. sch., then this gives a sheaf $T \mapsto G(T)$)

We have

$$h^1(G) - h^0(G) \leq \delta(G) - \alpha(G).$$

Def: An elementary adm. gp. sch. G is w/ $e(G) = 1$.

$-\ \mathbb{Z}[1/N]$	$\mathbb{Z}/p\mathbb{Z}$	μ_p
$-\ \mathbb{Z}$	$\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}^b$	μ_p, μ_p^b

Prop: We have the following table of mus

	$\mathbb{Z}/p\mathbb{Z}$	$(\mathbb{Z}/p\mathbb{Z})^D$	μ_p	μ_p^D
δ	0	1	0	1
α	1	1	0	0
h^0	1	0	ε'	0
h^1	0	0	ε	ε

$$\varepsilon' = \begin{cases} 0 & p \text{ odd} \\ 1 & p \text{ even} \end{cases}$$

$$\varepsilon = \begin{cases} 0 & p \text{ odd}, N \not\equiv 1 \ (p) \\ & \ldots\ p \text{ even } N \equiv 3 \ (4) \\ 1 & \text{otherwise} \end{cases}$$

$\Rightarrow$ (*)

- $0 \to G_1 \to G_2 \to G_3 \to 0$ SES of adm. gp. sch. $/\mathbb{Z}$ Suppose (*) ✓ for G_1 & G_3

$$\delta(G_2) = \delta(G_1) + \delta(G_3)$$
$$\alpha(G_2) = \alpha(G_1) + \alpha(G_3)$$

- LES

$$0 \to H^0(G_1) \to H^0(G_2) \to H^0(G_3) \to H^1(G_1) \to H^1(G_2) \to H^1(G_3) \to \underset{\to 0}{Ш}$$

$\Rightarrow h^1(G_2) - h^0(G_2) \le h^1(G_1) - h^0(G_1) + h^1(G_3) - h^0(G_3)$

$\square$