

MINIMAL REGULAR MODELS, JACOBIANS AND NÉRON MODELS

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1. INTRODUCTION AND MINIMAL MODELS

1.1. Motivating Example: Consider an elliptic curve E/\mathbb{Q}_p given by $y^2 = x^3 + ax + b$ (assume $p \neq 2, 3$). Then, one can naïvely reduce this curve to get $y^2 = x^3 + \bar{a}x + \bar{b}$. This reduced equation may or may not be well-defined as it depends on the choice of the equation. For example $\{y^2 = x^3 + p^6\} \cong \{y^2 = x^3 + 1\}/\mathbb{Q}_p$ (the isomorphism is given by $x' = p^2x, y' = p^3y$), but one reduces to $y^2 = x^3$ and the other to $y^2 = x^3 + 1$. It appears as though the first equation has a singularity, but it doesn't actually.

To resolve this ambiguity, we work with models. From this point until the end of the notes, we consider local field K with ring of integers \mathcal{O}_K , and residue field k . Let C/K be a “nice” (smooth, projective and geometrically integral) curve.

Any good notion of a model of C should remember the original curve C , and allow us to reduce points of C modulo \mathfrak{m} . In addition, there should be some overarching structure which allows us to utilise the full power of algebraic geometry. Both of these goals are achieved using \mathcal{O}_K -schemes. Since $\text{Spec}(\mathcal{O}_K)$ has two points, the zero ideal and the maximal ideal, any \mathcal{O}_K -scheme \mathcal{C} is composed of two fibres — the generic fibre \mathcal{C}_K , which is a curve over K , and the special fibre \mathcal{C}_k , which is a curve over k , and will be our reduction modulo \mathfrak{m} . We will demand that the former is isomorphic to C , but the latter can, within reason, be whatever we wants it to be.

Definition 1.1. A *model* of C/K is an \mathcal{O}_K -scheme \mathcal{C} that is of finite type, flat, proper, equipped with a specified morphism

$$\mathcal{C} \times_{\mathcal{O}_K} K \cong C.$$

We refer to $\mathcal{C} \times_{\mathcal{O}_K} K$ as the *generic fibre* of \mathcal{C} , and define its *special fibre* to be the k -scheme $\bar{\mathcal{C}} = \mathcal{C} \times_{\mathcal{O}_K} k$. Morphisms of models are morphisms of the underlying schemes which induce an isomorphism on the generic fibres.

Morally speaking, one can think of \mathcal{C} a surface that “parametrises” the curve C over $\text{Spec}(\mathcal{O}_K)$.

Example 1.2. Let E/\mathbb{Q}_p be an elliptic curve say with minimal Weierstrass equation $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Z}_p$. Then we can consider the scheme

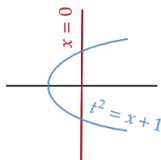
$$\mathcal{E} := \{y^2z - x^3 - axz^2 - bz^3 = 0\} \subset \mathbb{P}_{\mathbb{Z}_p}^2$$

which is a model of E . Its special fibre is the curve $\bar{E} = y^2 = x^3 + \bar{a}x + \bar{b}$ is well defined (up to isomorphism over \mathbb{F}_p). We will refer to this as the *Weierstrass model*. In the case that $v_K(\Delta) < 12$, we shall refer to such a model as a minimal Weierstrass model. In this case, the special fibre falls into one of three cases:

- \bar{E}/\mathbb{F}_p is an elliptic curve. This happens iff $\text{ord}_p(\Delta_E) = 0$ and is referred to as good reduction.
- \bar{E}/\mathbb{F}_p has a node. This happens iff the polynomial $x^3 + ax + b \pmod{p}$ has a unique double root, and is referred to as *multiplicative reduction*.
- \bar{E}/\mathbb{F}_p has a cusp. This happens iff the polynomial $x^3 + ax + b \pmod{p}$ has a triple root, and is referred to as *additive reduction*.

But how can we define other kinds of models for elliptic curves?

Example 1.3. Consider the elliptic curve given by $E : y^2 = x^3 + x^2 + p$. We proceed as follows. Substitute $y = tx$. Then, have $t^2x^2 = x^3 + x^2 + p$, which implies $p = x^2(t^2 - x - 1)$. This equation gives our new model of E . To look at the special fibre, we set $p = 0$ to obtain $x^2(t^2 - (x - 1)) = 0$, which looks like

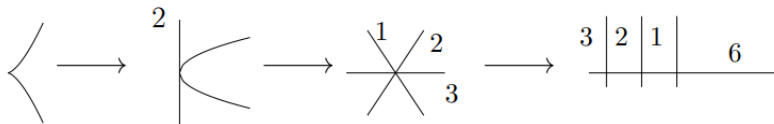


As the $x = 0$ line has multiplicity two, we can redraw our picture like this:



Thus, with this new equation, we resolved the singularity at $p = 0$ and have created a new model of E^1 .

Example 1.4. Now consider the elliptic curve given by $y^2 = x^3 + p$. We can keep blowing up the point $(0, 0)$ to resolve the singularities and obtain a sequence of models of E whose special fibres are shown below².

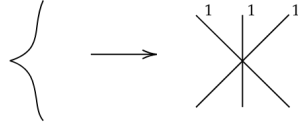


Example 1.5. Consider the elliptic curve given by $E : y^2 = x^3 + p^2$. Its models can be given as below³

¹Let $f(x, y, p) = y^2 - x^3 - x^2 - p = 0$. Then, $\partial f/\partial p = -1$, so the point $(0, 0, 0)$ is smooth, so we didn't really have to blow it up if we were seeking regular models, as the surface is already smooth. We did not resolve a singularity of the surface, only of the special fibre. What we built is actually called a strict normal crossing model. This example is just a dummy example to show the mechanics of how resolving singularities works, without dealing with some horrendous equations.

²Let $f(x, y, p) = y^2 - x^3 - p$. Then, $\partial f/\partial p = -1$, so again, we resolved the singularity on the special fibre and not the surface. In the diagram, the first model is a regular model and the fourth model is the strict normal crossings model.

³Note, if $f(x, y, p) = y^2 - x^3 - p^2$, then $\partial f/\partial p = 0$, if $p = 0$, so $(0, 0, 0)$ is a singularity of the surface. We can resolve this singularity by substituting $y = zx$ and $p = wx$ (note, we substitute both y and p , as opposed to just



These examples show that models are not required to be reduced (all components have multiplicity 1) or irreducible. It also illustrates one of the most effective tools we have to construct new models: blowing up points on the special fibre.

With infinite such options to go, we want to decide what kind of models we would like to work with. There are, broadly, two ways to go

- Insist that the scheme $\mathcal{C}/\mathcal{O}_K$ is regular.
- Ask that the special fibre be “not too bad” (aka “semistable”).

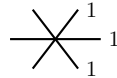
The first of these is always possible!

A model \mathcal{C} is regular if it is a model and a regular scheme. Among such models, there is a minimal such model. \mathcal{C} is *minimal* if for any other regular model \mathcal{C}' , there exists a map $\mathcal{C}' \rightarrow \mathcal{C}$ extending the identity over K .

Theorem 1.6. *A minimal regular model exists and is unique.*

Sadly, the special fibres of these minimal regular models can still be quite complicated.

Example 1.7. The curve $y^2 = x^3 + p^2$ has minimal regular model given by



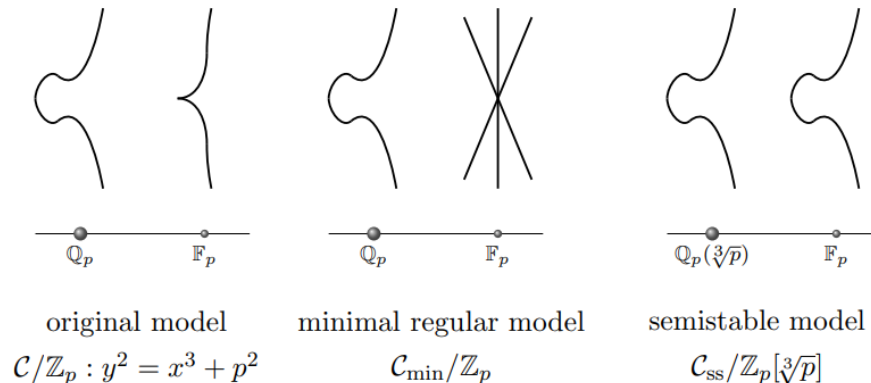
which has an ordinary triple point.

Definition 1.8. A curve is *semistable* if the only singularities are ordinary double points.

By contrast, semistable curves are all quite simple, however it is not always the case that a given curve has a semistable model. However,

Theorem 1.9 (Semistable Reduction Theorem). *There is always a finite extension K'/K where E achieves semistable reduction.*

Example 1.10. Let $E : y^2 = x^3 + p^2/\mathbb{Q}_p$. Then,



y). Then, we obtain $x^2(z^2 - x - w^2) = 0$. We remove the exceptional divisor x^2 to obtain a surface defined by $x = z^2 - w^2$, which is smooth, hence we have resolved our singularity with one blow up. The special fibre is given by $p = wx = 0 \implies w(z - w)(z + w) = 0$, and so we have three lines intersecting at our point, each with multiplicity 1. The model we just built is the regular model.

If K'/K sufficiently large, the reduction type of E/K' is constant and either good or multiplicative. In this first case, we say E has *potentially good reduction* (true iff and only if the j -invariant is integral). In the second, it has *potentially multiplicative reduction*.

Maybe you're wondering, hey, we haven't mentioned the group structure of an elliptic curve at all. You may have heard that an elliptic curve has genus 1, which means it is its own "Jacobian", which is an abelian variety. This means that we can define a notion of n -torsion.

Let $T_\ell(E) = \varinjlim_n E[\ell^n]$ be the Tate-module. Then

Theorem 1.11 (Néron-Ogg-Shafarevich). *Let $\ell \neq \text{char}(k)$ be a prime. E has good reduction if and only if $T_\ell(E)$ is unramified. E has semistable reduction if and only if inertia acts unipotently on $T_\ell(E)$.*

What about higher genus curves? A higher genus curve has no group structure. Without a group law, we have no Tate module, which means we can't do the arithmetic/Galois theory we need. So, to resolve this, we shall construct an analogue of the Tate module, the Jacobian, which is an abelian variety, so it has a group structure!

2. JACOBIANS

Okay, so why do we care about Jacobians? Initially, Jacobi introduced Jacobians in the analytic context – there is an analytic construction of the Jacobian, but I will be skipping it in this talk (would highly recommend giving it a read though!). In the number theory context, Weil introduced Jacobians when he was trying to prove the Weil conjectures. If you try and prove the Weil conjectures for elliptic curves, the key point is that you need to look at the Frobenius action on p -torsion (really, the p -adic Tate module). But if you want to prove Weil conjectures for curves, what on earth do you do?

And so, Weil introduced the Jacobian, because it also has a notion of p -torsion, and then we can do the whole theory there, as the curve embeds into the Jacobian – and so the curve doesn't have p -torsion, but the Jacobian does. But what is the Jacobian? We're first going to show a general algebraic construction/existence using functors, and then give a more explicit construction over curves.

2.1. Functorial description of $\text{Jac}(X)$.

Definition 2.1. Let X be a projective curve. $\text{Pic}(X)$ is the group of isomorphism classes of line bundles on X . We let $\text{Pic}^0(X) \subset \text{Pic}(X)$ be the subgroup of degree 0 line bundles.

We would like to give Pic^0 the structure of a variety. The way we're going to do this is to define Pic^0 by its functor of points. Why would we want to do such a thing? We want it to be a variety so it makes sense do things such as "reduce modulo p ".

So, the goal is to define the functor of points for the Jacobian, $\text{Pic}^0(X)$, over a base scheme T .

Let us start by defining a naïve functor $F(T)$ as the set of isomorphism classes of line bundles L on $X_T = X \times T$ that have degree 0 on each fibre $X \times \{t\}$.

The first problem we encounter is that F cannot be represented by a scheme J . If it were, any line bundle on T pulled back via the projection $p : X_T \rightarrow T$ would give a bundle $p^*(L)$ that is trivial on each fibre. By the universal property, this would be classified by a constant map $f : T \rightarrow J$, which incorrectly forces $p^*(L)$ to be the trivial bundle overall, creating a contradiction.

The way we could fix this is that we quotient out the problematic pullbacks from the base. We define a new functor $G(T) := F(T)/p^*\text{Pic}(T)$.

Unfortunately, even after quotienting out the base line bundles, $G(T)$ is still not representable because it fails to be a sheaf (it lacks good descent properties).

The next best solution is to rigidify. The fundamental reason G fails to be a sheaf is the existence of non-trivial automorphisms (scaling by constants). To solve this, we assume X has a rational

point $x \in X(k)$, and we define $\mathcal{G}_x(T)$ as the category of pairs (L, i) , where L is a fibrewise degree 0 line bundle on $X \times T$, and $i : L|_{\{x\} \times T} \xrightarrow{\sim} \mathcal{O}_T$ is an explicit isomorphism trivialising the bundle along the section x . The trivialization i forces any automorphism to scale strictly by 1, and G_x perfectly satisfies descent (it is a sheaf).

One can easily prove that $G_x(T) \rightarrow G(T)$ is an isomorphism. Therefore, the presence of a rational point guarantees that G is a sheaf and is representable. This representing scheme is exactly $Jac(X)$, the Jacobian of X .

2.2. Explicit construction of $Jac(X)$. We trade line bundles for effective divisors. We have a curve X and we want to build a group out of it. If we have an elliptic curve, looking at points works. If we have a curve of genus g , we will need g unordered points. So, we use effective divisors of degree g . Let us consider the symmetric power space $X^{(g)}$, and let D and D' be two degree g divisors. We want to add them, and get a third divisor D'' . We would want this to be of degree g too, so we subtract g copies of the rational point $x \in X$ to get a divisor of degree g , i.e., $D + D' - g[x]$. Let $\ell(D)$ be the dimension of the vector space of rational functions associated to a divisor D , i.e., if you have a divisor $D = P + Q$, we want to find functions that have poles at P and Q (and nowhere else), but are otherwise well-behaved. If $\ell(D) = 0$, no such function exists, and if $\ell(D) = 1$, there is exactly one such function (up to scaling). Riemann Roch gives us an inequality $\ell(D) \geq \deg(D) - g + 1$. If $\ell(D) = 0$, no such function exists. Because we engineer our divisor to have degree g , Riemann Roch guarantee that $\ell(D)$ must have dimension at least 1.

Now, an appropriate semi-continuity result tells us that the set

$$U := \{(D, D') \in X^{(g)} \times X^{(g)} : \ell(D + D' - g[x]) = 1\},$$

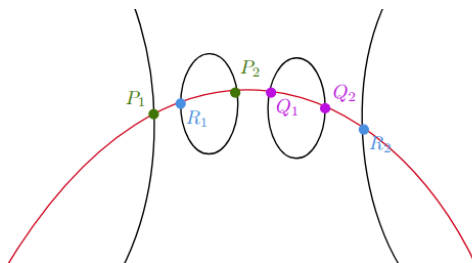
is open. One can show it is non-empty, and thus dense. So, we have a rational map $X^{(g)} \times X^{(g)} \rightarrow X^{(g)}$.

Weil⁴ then proved a general result saying that such a rational group law uniquely extends to an actual group law, i.e. there exists a unique group variety J with a birational group homomorphism $X^{(g)} \rightarrow J$. Finally, one shows that J actually represents G .

Sometimes, we can explicitly determine what this function f guaranteed by Riemann-Roch is.

Example 2.2. *For Genus 1 (Elliptic Curves):* $X^{(1)}$ is just one point, and so this f is just the line between 2 points, which intersects the curve at a third point, as expected.

Example 2.3. *For Genus 2 (Hyperelliptic Curves):* $X^{(2)}$ means our inputs are pairs of points: $D = P_1 + P_2$ and $D' = Q_1 + Q_2$. It turns out that this f is a cubic, that cuts our curve at two other points.



⁴Here are some quotes from Grothendieck to Serre about Weil's construction:

"I have read the statements in Weil's books on abelian varieties, in the hope that the proofs might have been improved since he wrote them; they are really discouraging in Weil, and on top of that his language disgusts me."

"I am sad to hear that one cannot present Weil's results without juggling with generic points; as a matter of fact, the unbridled abuse of generic points necessarily hides the few situations in which their use is truly essential, such as the proof that every endomorphism of the Jacobian comes from a correspondence. As Chevalley says, one feels frustrated when faced with a proof like that one."

3. NÉRON MODELS

But now we have a new problem. To understand the reduction of Jacobians, we can't use curve tools. We need abstract group scheme tools. We need a model over \mathcal{O}_K where rational points uniquely extend.

As mentioned before, the Néron model of an elliptic curve is the smooth locus of its minimal regular model. This sadly does not generalise to higher dimension abelian varieties, so we shall give a functorial definition of the same.

Theorem 3.1. *Let A/K be an abelian variety. Then there exists a smooth, separated, finite type group scheme $\mathcal{A}/\mathcal{O}_K$ with generic fibre A , satisfying the universal property (the Néron mapping property):*

for each smooth \mathcal{O}_K -scheme \mathcal{Y} , any K -morphism $\mathcal{Y}_K \rightarrow A$ extends uniquely to an \mathcal{O}_K -morphism $\mathcal{Y} \rightarrow \mathcal{A}$. We call \mathcal{A} the Néron model of A .

Facts and remarks:

- Note that, unlike for curves, we have dropped “properness” in favour of “smoothness”. This is because the Néron mapping property specifies the functor of points of \mathcal{A} (uniquely up to isomorphism), but only on smooth schemes, and if the Néron model wasn't smooth, Yoneda's lemma wouldn't apply.
- The mapping property implies $\mathrm{Hom}_{\mathcal{O}_K}(\mathcal{Y}, \mathcal{A}) \cong \mathrm{Hom}_K(\mathcal{Y}_K, A)$. So, if $\mathcal{Y} = \mathrm{Spec}(\mathcal{O}_K)$, we get $\mathcal{A}(\mathcal{O}_K) = A(K)$.
- Néron models do not generally commute with base change; if you pass to a ramified field extension, you usually have to build a new Néron model to capture all the new rational points. More precisely, let K'/K is a finite extension, and $\mathcal{A}, \mathcal{A}'$ be Néron models of A/K and A/K' respectively. Then $\mathcal{A} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ is a smooth group scheme with generic fibre isomorphic to $A_{K'}$, but the map

$$f : \mathcal{A} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \rightarrow \mathcal{A}'$$

is not necessarily an isomorphism. It is only an isomorphism if K'/K is unramified.

Definition 3.2. The *reduction* of an abelian variety over K is the group variety $\mathcal{A}_k = \mathcal{A} \times_{\mathcal{O}_K} k$ over k . If this is an abelian variety then we say that A/K has *good reduction*. We define \mathcal{A}_k^0 be the *identity component* of \mathcal{A}_k .

The Néron mapping property gives us $A(K) = \mathcal{A}(\mathcal{O}_K)$, so we have reduction homomorphism $A(K) \rightarrow \mathcal{A}_k(k)$. We write $A_0(K)$ for the points in $A(K)$ reducing to $\mathcal{A}_k^0(k)$. The group $A(K)/A_0(K)$ is finite the its cardinality is called the *Tamagawa number*.

Alternatively, one may define a finite étale group scheme over k

$$\Phi = \mathcal{A}_k / \mathcal{A}_k^0,$$

and we have an exact sequence

$$0 \longrightarrow \mathcal{A}_k^0 \longrightarrow \mathcal{A} \longrightarrow \Phi \longrightarrow 0$$

and Φ is called the *group of connected components* of A .

Chevalley's structure theorem gives us a description of \mathcal{A}_k^0 if our base field k is perfect. We have another exact sequence given by

$$0 \longrightarrow T \times U \longrightarrow \mathcal{A}_k^0 \longrightarrow B \longrightarrow 0$$

where

- B is a k -abelian variety.
- T is an algebraic torus.

- U is a unipotent algebraic group.

Definition 3.3. We say A has

- *good reduction* if U, T are trivial.
- *semistable reduction* if U is trivial.
- *(purely) toric reduction* if U, B are trivial.
- *split toric reduction* if U and B are trivial and T is a split torus over k .

Theorem 3.4 (Neron-Ogg-Shafarevich, Grothendieck). *Let $\ell \neq \text{char}(k)$.*

A has good reduction if and only if $T_\ell(A)$ is an unramified representation of $\text{Gal}(\bar{K}/K)$.

A has semistable reduction if and only if I_K acts unipotently on $T_\ell(A)$