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(I) Hecke Correspondences

- we want to define Hecke operators in terms of correspondences

Definition (Correspondence)
A correspondence is a diagram of the form

$$\begin{tikzcd} & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

where X, Y smooth projective curves; p_1, p_2 finite maps

- a correspondence $f: X \dashrightarrow X$ induces a map $H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{Z})$ via $(p_2)_* p_1^*$
($(p_2)_*$ adjoint to p_2^* under cup product pairing
↳ f also induces a map $H^0(X, \Omega^1) \to H^0(X, \Omega^1)$ by $(p_2)_* p_1^*$ (diff. forms)
which is compatible with Hodge decomposition
$H^1(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^0(X, \Omega^1) \oplus \overline{H^0(X, \Omega^1)}$
- correspondences also act on divisors; explicitly, for $x \in X$
$f_*([x]) := \sum_{p_1(y)=x} [p_2(y)]$
↳ this ~~factors~~ ~~through~~ preserves principal divisors & thus induces a map on the Jacobian $f_*: \mathrm{Jac}(X) \to \mathrm{Jac}(X)$
given by $f_* = (p_2^*)^\vee p_1^*$ where $p_i^*: \mathrm{Jac}(X) \to \mathrm{Jac}(Y)$ natural map and $(-)^\vee$ dual isogeny

- recall that $X_0(N) = \{(E, C) : E$ elliptic curve, C cyclic subgroup of order N of $E\}/\sim$
- we define two degeneracy maps $p_1(E, C) = (E/C_p, C/C_p)$
where C_p subgroup of C of order p and $p_2(E, C) = (E, C_N)$
C_N subgroup of C of order N

$\Rightarrow$
$$\begin{tikzcd} & X_0(Np) \arrow[dl, "{p_1: (E,C) \mapsto (E/C_p, C/C_p)}"'] \arrow[dr, "{p_2: (E,C) \mapsto (E, C_N)}"] & \\ X_0(N) & & X_0(N) \end{tikzcd}$$

gives rise to correspondence T_p

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$T_p(E,C) = \sum p_2(E',C')$
$p_1(E',C') = (E,C)$

assume $p \nmid N$

- we can describe this explicitly: ~~[illegible]~~
take $(E,C) \in Y_0(N)$; a point above (E,C) in $p_1^{-1}(E,C)$ is of the form (E',C') where $E' = E/D$ for $D \subset E[p]$ cyclic of order p and $C' = \frac{C+D}{D} + C_p'$ where $C_p' = E[p]/D$.
Then $p_2(E',C') = (C'/C_p', C'/C_p') = ((E/D)/(E[p]/D), C'/C_p')$
Now $(E/D)/(E[p]/D) \cong E/E[p] \cong E$ via multiplication by p
and $C'/C_p' \cong (\frac{C+D}{D} + C_p')/C_p' \cong \frac{C+D}{D}/(C_p' \cap \frac{C+D}{D}) \cong \frac{C+D}{D} \cong C/C \cap D \cong C$
Hence $T_p(E,C) = \sum_{\substack{D \subset E[p] \\ \text{order } p}} (E/D, \frac{C+D}{D})$

for $p \mid N$, we get $U_p(E,C) = \sum_{\substack{D \subset E[p] \\ D \not\subset C}} (E/D, \frac{C+D}{D})$

- this is compatible with the double coset description of the Hecke operators

(II) Structure of the Hecke Algebra

- define $\tilde{\mathbb{T}} := \mathbb{Z}[T_p : p \nmid N]$ and $\mathbb{T} := \mathrm{im}(\tilde{\mathbb{T}} \to \mathrm{End}\, S_2(\Gamma_0(N)))$
  $\hookrightarrow \mathbb{T}_{\mathbb{Q}}$ $\mathbb{Q}$-span
- Petersson inner product: for $f, g \in S_2(\Gamma_0(N))$, $f(z)dz \wedge \overline{g(z)dz}$ $= 2i f(z)\overline{g(z)}\, dx\, dy$ $\Gamma_0(N)$-invariant 2-form
  $\Rightarrow \langle f, g \rangle := \int_{H/\Gamma_0(N)} f(z)\overline{g(z)}\, dx\, dy$ converges & $\langle \cdot,\cdot \rangle$ positive definite Hermitian inner product on $S_2(\Gamma_0(N))$

(i) T_p is self-adjoint (so the T_p on $S_2(\Gamma_0(N))$ can be simultaneously diagonalised)
(ii) there is a basis of $S_2(\Gamma_0(N))$ of simultaneous eigenvectors (i.e. Hecke eigenforms)
(iii) $\mathbb{T}_{\mathbb{C}}$ semi-simple $\mathbb{C}$-algebra

Theorem (Multiplicity 1)
Suppose that N is prime. Let f, g be two normalised eigenforms with the same system of eigenvalues. Then $f = g$.

<u>Proof</u> We know $a_p(f) = a_p(g)$ for all $p \nmid N$. By multiplicativity and recurrence formulas for T_p, we get $a_n(f) = a_n(g)$ for $(n,N)=1$
$\hookrightarrow T_{p^{n+1}} = T_p T_{p^n} - p T_{p^{n-1}}$
Set $h = f-g$, so $a_n(h) = 0$ for $N \nmid n$ (N prime) $\Rightarrow h = \sum_{n \geq 1} a_{nN} q^{nN}$
$\Rightarrow h(z + \frac{1}{N}) = h(z) \Rightarrow (h|_2 [\gamma](z)) = (cz+d)^{-2} h(\gamma z) = h$
for $\gamma \in \Gamma_0(N)$ or $\gamma = \begin{pmatrix} 1 & 1/N \\ 0 & 1 \end{pmatrix}$ which together generate $\sigma^{-1} \Gamma(1) \sigma$
where $\sigma = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow h|_2[\sigma]$ invariant under $\Gamma(1) \Rightarrow h|_2[\sigma] \in S_2(\Gamma(1)) = 0$
$\Rightarrow h = 0 \Rightarrow f = g$

- define old subspace $S_2(\Gamma_0(N))^{old} = \mathrm{span}\{f(z), f(pz) : f \in S_2(\Gamma_0(\frac{N}{p})) : p|N\}$
and $S_2(\Gamma_0(N))^{new}$ its orthogonal complement
$\Rightarrow$ Multiplicity 1 holds on newspace for N not prime also

(iv) for $S_2(\Gamma_0(N)) = \bigoplus_\alpha S_2(\Gamma_0(N))_\alpha$ where $S_2(\Gamma_0(N))_\alpha := \{f : Tf = \alpha(T) f \ \forall T \in \mathbb{T}\}$
we have $\dim S_2(\Gamma_0(N))_\alpha = 1$
(v) there is a bijection between homomorphisms $\mathbb{T} \to \mathbb{C}$ and normalised cusp forms
(vi) $S_2(\Gamma_0(N))$ is free of rank 1 as a $\mathbb{T}_\mathbb{C}$-module

- by the construction of the Hecke operators as correspondences, $\hat{\mathbb{T}}$ acts on $H^1(X_0(N), \mathbb{Z})$ and the isomorphism is compatible with the actions on each side $\boxed{H^1(X_0(N), \mathbb{Z}) \otimes \mathbb{C} \cong S_2(\Gamma_0(N)) \oplus \overline{S_2(\Gamma_0(N))}}$
(here we identify $S_2(\Gamma_0(N))$ with $H^0(X_0(N), \Omega^1)$)
- if $T \in \hat{\mathbb{T}}$ acts by zero on $S_2(\Gamma_0(N))$, then it does so on $\overline{S_2(\Gamma_0(N))}$ aswell and hence also on $H^1(X_0(N), \mathbb{Z})$
$\Rightarrow$ the image of $\hat{\mathbb{T}}$ in $\mathrm{End}(H^1(X_0(N), \mathbb{Z}))$ is isomorphic to $\mathbb{T}$
- as $\mathrm{End}_\mathbb{Z}(H^1(X_0(N), \mathbb{Z})) \cong M_{2g}(\mathbb{Z})$, g genus of $X_0(N)$, we have
(a) $\mathbb{T}$ is a finite rank free $\mathbb{Z}$-module
(b) the Hecke eigenvalues of a normalised cusp eigenform are algebraic integers
(c) $\mathbb{T} \otimes \mathbb{Q}$ is a product of finitely many number fields
<u>Proof</u>: $S_2(\Gamma_0(N))$ free of rank 1 over $\mathbb{T}_\mathbb{C} \Rightarrow H^1(X_0(N), \mathbb{Z}) \otimes \mathbb{C}$ free of rank 2 over $\mathbb{C}$; now $H^1(X_0(N), \mathbb{Q})$ module over semi-simple $\mathbb{T}_\mathbb{Q}$ $\leftarrow$ free rk 2 when $\otimes \mathbb{C}$

(III) The Atkin-Lehner Involution

- the space $X_0(N)$ admits a natural involution

Definition (Atkin-Lehner involution)
$w : X_0(N) \to X_0(N)$ given by $(E, C) \mapsto (E/C, E[N]/C)$ on points
↳ if we consider $X_0(N) = \{ [E \xrightarrow{\varphi} E'] : \deg \varphi = N \} / \cong$ then it sends a degree N isogeny to its dual
$\Rightarrow w^2 = \mathrm{id}$

- this involution w induces an action on $S_2(\Gamma_0(N)) \cong H^0(X_0(N), \Omega^1)$ by pullback
↳ one can compute $(wf)(z) = f\left(-\frac{1}{Nz}\right)$
- Fact: w commutes with T_p for $p \nmid N$ and thus preserves eigenspaces
$\Rightarrow$ if f is an eigenform then $wf = \pm f$ because the eigenspaces are one-dimensional
- w also induces an involution w_* on $J_0(N)$

(IV) The Eichler-Shimura Relation

- we are now interested in the Hecke correspondence mod p ⇝ integral model
- recall: $\mathcal{F}_{\Gamma_0(N)}$: Schemes/$\mathbb{Z}[\frac{1}{N}] \to$ Sets
$S \mapsto \{$elliptic curves over S + cyclic subgroup $< E[N]$ of order $N\} / \cong$

representable by a smooth Deligne-Mumford stack $\mathcal{M}_0(N)$ over $\mathbb{Z}[\frac{1}{N}]$
↳ compactification $\overline{\mathcal{M}}_0(N)$ parametrises generalised elliptic curves over $\mathbb{Z}$ with finite flat group scheme of N that in each fibre over $\mathbb{Z}$ meets each irreducible component of E
- let $M_0(N)$ be its coarse space (so $M_0(N)_{\mathbb{Q}} \cong X_0(N)$)
- define $\tilde{f} : \mathcal{M}_0(Np) \to \mathcal{M}_0(N)$, $(E, C_N, C_p) \mapsto (E, C_N)$
where C_N level($\Gamma_0(N)$)-& C_p level($\Gamma_0(p)$)-structure; this descends to a map $f : M_0(Np) \to M_0(N)$ on the coarse spaces
- similarly, define $\tilde{g} : \mathcal{M}_0(Np) \to \mathcal{M}_0(N)$, $(E, C_N, C_p) \mapsto (E/C_p, \frac{C_N + C_p}{C_p})$ and g on coarse spaces.
↳ this is composition of $\tilde{f}$ with Atkin-Lehner ω_p

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relative divisor D of $X/\mathcal{O}$: closed subscheme $D \subset X$ such that $D \to \operatorname{Spec} \mathcal{O}$ flat and for every $s \in \operatorname{Spec} \mathcal{O}$, $D_s = D \times_{\mathcal{O}} k(s)$ effective Cartier divisor on X_s
(effective cartier divisor : ideal sheaf I_D invertible $\mathcal{O}_X$-module)

the correspondence (f, g finite flat)

$$\begin{tikzcd} & M_0(Np) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ M_0(N) & & M_0(N) \end{tikzcd}$$

recovers the Hecke correspondence T_p on the generic fibre (i.e. over $\mathbb{Q}$).

Lemma

$\mathcal{O}$ complete DVR, $K = \operatorname{Frac}(\mathcal{O})$, residue field k, $X/\mathcal{O}$ proper smooth curve, $f, g : Y \rightrightarrows X$ finite flat maps giving a correspondence on X, $J_K = \operatorname{Jac}(X_K)$ and $h_K : J_K \to J_K$ induced by (f, g), $J/\mathcal{O}$ Néron model of J, $h : J \to J$ extension of h_K, $h_k : J_k \to J_k$ base change to k, D_0 degree zero divisor on X_k defining a point $x_0 \in J(k)$. Then $h_k(x_0)$ is represented by $g_*(f^*(D_0))$ in J_k

Proof

Lift D_0 to a relative divisor D on X (possible since X is smooth over $\mathcal{O}$)
Let $D' = g_*(f^*(D))$, which is again a relative divisor on X. Then D, D' define points $x, y \in J(\mathcal{O})$. Then $h(x)$ is the unique $\mathcal{O}$-point of J extending the K-point $h_K(x_K)$.
By definition, $h_K(x_K)$ is represented by $g_*(f^*(D_K))$, thus equal to y_K
As y extends y_K, $h(x) = y \Rightarrow h(x_0) = y_0$
· y_0 is represented by $g_*(f^*(D_0))$ as operations commute with base change

Corollary

T_p is computed by $g_* f^*$ on $J_0(N)_{\mathbb{F}_p}$

Theorem (Eichler – Shimura)

we have $T_p = F + V$ on $J_0(N)_{\mathbb{F}_p}$ where F is the Frobenius and V is the Verschiebung map

Proof (we work in char p)

- $\tilde{i} : M_0(N) \to M_0(Np)$, $\tilde{i}(E) = (E, \ker F)$ where $F : E \to E^{(p)}$ Frobenius
- $\tilde{j} : M_0(N) \to M_0(Np)$, $\tilde{j}(E) = (E^{(p)}, \ker V)$ = comp. of $\tilde{i}$ and Atkin-Lehner w_p and $V : E^{(p)} \to E$ Verschiebung

· these induce maps i, j on $M_0(N)$ and we have
$fi = id$, $gj = id$, $fj = F$, $gi = F$; so i, j are closed immersions
(for example $f(i(E,C_N)) = f(E, C_N, \ker F) = (E, C_N)$ etc.)

· $M_0(N)^{ord}$ open subscheme of $M_0(N)$ ~~$_{\mathbb{F}_p}$~~, ordinary locus
· we have a map $(i,j): M_0(N)^{ord} \sqcup M_0(N)^{ord} \longrightarrow M_0(Np)^{ord}$
· think of a point in $M_0(Np)^{ord}(k)$ as (E, C_N, C_p) where E/k ordinary, $C_N \subset E[N]$ order N, $C_p \subset E[p] \cong \mu_p \oplus \mathbb{Z}/p\mathbb{Z}$ order p
⟹ ordinary E has exactly two subgroup schemes of order p
(i) C_p connected ($\cong \mu_p$) ⟹ $C_p = \ker F$ ⟹ $(E, C_N, C_p) = i(E, C_N)$
(ii) C_p étale ($\cong \mathbb{Z}/p\mathbb{Z}$) ⟹ $C_p \cong \ker V$
↳ set $q: E \to E/C_p =: E_0$, $C_0 = q(C_N)$ and note $E \cong E_0^{(p)}$ & $q \cong V$ under this
⟹ $(E, C_N, C_p) = (E_0^{(p)}, V^{-1}(C_0), \ker V) = j(E_0, C_0)$
⟹ (i,j) isomorphism!
consider Hecke correspondence:
(by above)

$$\begin{tikzcd}
& M_0(N)^{ord} \arrow[dl, "id"'] \arrow[drr, "F"] & \sqcup & M_0(N)^{ord} \arrow[dlll, "F"'] \arrow[dr, "id"] & \\
\underbrace{M_0(N)^{ord}}_{f} & & & & \underbrace{M_0(N)^{ord}}_{g}
\end{tikzcd}$$

with i on the first copy and j on the second.

· hence on ~~T_p~~ $M_0(N)^{ord}$, the Hecke correspondence T_p is a disjoint union of two correspondences
$T_p = (id, F) + (F, id)$ (on divisors on $M_0(N)^{ord}$, it acts by addition)
· we have $(id, F)(D) = F_* id^* D = F_* D = F(D)$ and
$(F, id)(D) = id_* F^* D = F^* D = V(D)$
⟹ $T_p = F + V$ holds on points in $J_0(N)$ represented by divisors in the ordinary locus
· there are only finitely many points in the supersingular locus, so any divisor class is linearly equivalent to one supported in the ordinary locus ⟹ $T_p = F + V$ on $J_0(N)_{\mathbb{F}_p}$

(V) Galois Representations

Theorem

There exists a unique (up to isomorphism) semisimple representation $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\bar{\mathbb{Q}}_\ell)$ such that

(i) ρ unramified away from $N\ell$

(ii) $\mathrm{tr}(\rho(\mathrm{Frob}_p)) = a_p$ for $p \nmid N\ell$

(iii) $\det \rho = \chi_\ell$

for $f = \sum_{n \geq 1} a_n q^n$ a normalised ~~weig~~ eigenform in $S_2(\Gamma_0(N))$

The Shimura Construction

Fix (prime) N, normalised weight 2 cusp form $f \in S_2(\Gamma_0(N))$

$\alpha : \mathbb{T} \to \mathbb{C}$ homomorphism giving the eigenvalues of f, i.e. $T_p f = \alpha(T_p) f$

Then $K = \alpha(\mathbb{T} \otimes \mathbb{Q})$ is a number field and $\mathcal{O} = \alpha(\mathbb{T})$ $\quad = a_p(f)$

an order in K, $\underline{a} := \ker \alpha \subset \mathbb{T}$

- set $A_f := J_0(N) / \underline{a} J_0(N)$ where $\underline{a} J_0(N) := \sum_{T \in \underline{a}} T J_0(N)$ (abelian subvariety of $J_0(N)$)
- we prove properties of this A_f, which will translate to the required properties of ρ_f

(i) Dimension

- tangent space to $J_0(N)$ at identity: $T_0(J_0(N)) \overset{\text{see duality } (*)}{\cong} H^1(X_0(N), \Omega^1)$

over $\mathbb{C}$ $\cong S_2(\Gamma_0(N))$, which is a free module of rank 1 over $\mathbb{T}_{\mathbb{C}}$

$\Rightarrow T_0(J_0(N))$ free module of rank 1 over $\mathbb{T}_{\mathbb{Q}}$

$(*)$: $T_0 \mathrm{Pic}^0 X \cong H^1(X, \mathcal{O}_X) \Rightarrow H^1(X, \mathcal{O}_X)^\vee \cong H^0(X, \Omega^1)$

$\Rightarrow T_0(J_0(N)) / \underline{a}\, T_0(J_0(N)) = T_0(A_f)$ one-dimensional vector space of K $\Rightarrow$ $A_f \overset{\text{Proposition}}{\text{is}}$ an abelian variety of dimension $[K : \mathbb{Q}]$

$(T_0 A_f \cong \mathbb{T}_{\mathbb{Q}} / \underline{a}\, \mathbb{T}_{\mathbb{Q}} \cong K)$

$\Rightarrow \dim_{\mathbb{Q}} T_0 A_f = [K : \mathbb{Q}]$

(ii) Good Reduction

Lemma

Let B be an abelian variety over a DVR with good reduction and let A be a subquotient of B. Then A has good reduction.

Proof

ℓ prime different from residue characteristic

B good reduction $\Rightarrow V_\ell(B)$ unramified representation $\overset{\text{NOS}}{\Leftarrow\Rightarrow}$ by Néron-Ogg-Shafarevich

$V_\ell(A)$ subquotient of $V_\ell(B)$, so also unramified $\overset{\text{NOS}}{\Longrightarrow}$ A good reduction

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- $X_0(N)$ has smooth proper integral model over $\mathbb{Z}[\frac{1}{N}]$, say $\mathcal{X}_0(N) \to \operatorname{Spec} \mathbb{Z}[\frac{1}{N}]$ ⇒ take its relative Picard scheme $\operatorname{Pic}^0_{\mathcal{X}_0(N)/\mathbb{Z}[\frac{1}{N}]}$ which is abelian scheme over $\mathbb{Z}[\frac{1}{N}]$ and its generic fibre is $J_0(N)$
⇒ $J_0(N)$ has good reduction at primes outside N

Proposition
A_f has good reduction away from N

(iii) Structure of the Tate Module
Let V_ℓ be the rational Tate module $T_\ell(J_0(N)_{\mathbb{F}_p})[\frac{1}{\ell}]$, a $\mathbb{Q}_\ell$-vector space
- as $\mathbb{T}$ acts on $J_0(N)$, V_ℓ is a $\mathbb{T}_{\mathbb{Q}_\ell}$-module which is free of rank 2
- Frobenius F acts on V_ℓ and commutes with action of $\mathbb{T}$, so can regard action on F as matrix in $GL_2(\mathbb{T}_{\mathbb{Q}_\ell})$

Proposition
We have $tr(F|V_\ell) = T_p$ and $det(F|V_\ell) = p$

Proof
Let $\langle \cdot,\cdot \rangle : V_\ell \times V_\ell \to \mathbb{Q}_\ell(1)$ be the Weil pairing on V_ℓ and the T_q for $q \nmid N$ are self-adjoint with respect to this pairing
(T_q for $q \nmid N$ induced by (f_ℓ, g_ℓ) & adjoint of T_ℓ induced by transposed correspondence (g_ℓ, f_ℓ) ⇒ these are equal: sum over p-isogenies $E \to E'$ vs. sum over p-isogenies $E' \to E$, which are in bijection via dual isogeny)
⇒ upshot: $\varphi : V_\ell \to V_\ell^*$, $x \mapsto \langle -, x \rangle$ isomorphism of $\mathbb{T}_{\mathbb{Q}_\ell}$-modules
(as Weil pairing perfect & $T\varphi(x)(y) = \varphi(Tx)(y) = \langle y, Tx \rangle = \langle Ty, x \rangle = \varphi(x)(Ty)$)
- we also have $\varphi(Fx) = V\varphi(x)$: $\varphi(Fx)(y) = \langle y, Fx \rangle = \frac{1}{p}\langle FVy, Fx \rangle = \langle Vy, x \rangle$
$= \varphi(x)(Vy) = (V\varphi(x))(y)$ $\qquad \langle Fa, Fb \rangle = p\langle a,b \rangle$
⇒ $V \circ \varphi = \varphi \circ F$ ⇒ $V = \varphi \circ F \circ \varphi^{-1}$ on V_ℓ^* ⇒ $tr(F|V_\ell) = tr(V|V_\ell^*)$
- matrix for V on V_ℓ^* just transpose for matrix of V on V_ℓ
⇒ $Tr(V|V_\ell^*) = Tr(V|V_\ell)$ ⇒ $tr(F|V_\ell) = tr(V|V_\ell)$
- by Eichler-Shimura ($T_p = F + V$), $Tr(T_p) = 2T_p$ as V_ℓ free of rank 2 over $\mathbb{T}_{\mathbb{Q}_\ell}$ and $Tr(T_p) = 2Tr(F)$ ⇒ $tr(F) = T_p$

$V_\ell \cong \mathbb{T}_{\mathbb{Q}_\ell}^{\oplus 2}$ and $T_p \in \mathbb{T}_{\mathbb{Q}_\ell}$ acts by scalar mult.

- multiply Eichler-Shimura by F: $T_pF = F^2 + VF \iff F^2 - T_pF + p = 0$
- $det(F|V_\ell) = p$ follows

· we have natural map $\sigma = \mathbb{T}/\underline{a} \to \mathrm{End}(A_f)$ => consider $V_\ell(A_f)$ as free rank 2 $K \otimes \mathbb{Q}_\ell$ - module

Proposition

Pick a prime $p \nmid N$ and let $F_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the Frobenius at p. Then $\mathrm{tr}(F_p | V_\ell(A_f)) = a_p$ and $\det(F_p | V_\ell(A_f)) = p$.

Proof

This holds for $V_\ell(J_0(N))$ by previous proposition.

Now $\pi: J_0(N) \to A_f$ quotient map is compatible with Hecke action: if $T \in \underline{a}$ then $T(J) \subset \underline{a}J$, so $\pi \circ T = 0$ => action of $\mathbb{T}$ on $J_0(N)$ descends to an action of $\mathbb{T}/\underline{a}$ on A_f (and π defined over $\mathbb{Q}$, so commutes with F_p)

Now T_p acts by $\alpha(T_p) = a_p(f)$ (as $\mathbb{T}/\underline{a} = \mathrm{im}(\alpha)$), so $\mathrm{tr}(F_p | V_\ell(A_f)) = a_p$; det stays p

Proof of Theorem fix embedding $K \hookrightarrow \overline{\mathbb{Q}}_\ell$

(i) Existence: $K \hookrightarrow \overline{\mathbb{Q}}_\ell$ determines a place of K above ℓ, i.e. an idempotent e in $K \otimes \mathbb{Q}_\ell$ ($K \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \cong \prod_{\tau: K \hookrightarrow \overline{\mathbb{Q}}_\ell} \overline{\mathbb{Q}}_\ell$, $e = (0, 0, \dots, 1_\tau, 0, \dots, 0)$)

=> take ρ to be the semi-simplification of $e(V_\ell(A_f))$

· by properties proven for A_f, ρ unramified away from $N\ell$, $\mathrm{tr}(\rho(\mathrm{Frob}_p)) = a_p$ and $\det(\rho(F_p)) = p$ for $p \nmid N\ell$, so $\det \rho = \chi_\ell$ chebotarev

(ii) Uniqueness: suppose ρ' also satisfies 3 properties in Theorem so $\mathrm{tr}(\rho(F_p)) = \mathrm{tr}(\rho'(F_p))$ for all $p \nmid \ell N$ => by Chebotarev, $\mathrm{tr}(\rho(g)) = \mathrm{tr}(\rho'(g))$ for all $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ => $\rho \cong \rho'$ by Brauer-Nesbitt & semisimple

$k > 2$: modular forms are not just differentials on $X_0(N)$ but sections of powers of the Hodge bundle

=> Deligne uses local system / sheaf V_{k-2} on modular curve, built roughly from $\mathrm{Sym}^{k-2} T_\ell(\mathcal{E})$ where $\mathcal{E}$ universal elliptic curve

=> then one studies $H^1_{et}(Y_0(N)_{\overline{\mathbb{Q}}}, V_{k-2})$; Hecke operators act on this cohomology and the f-eigenspace is 2-dimensional

· Galois also acts on this and commutes with Hecke action and we get $\rho_f: G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_\ell)$ but now $X^2 - a_p(f) X + \varepsilon(p) p^{k-1}$, $p \nmid N\ell$ char. poly.